Functional interpretation of arithmetical comprehension

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Full comprehension

Spector studied the full comprehension schema in type zero:

$$\mathrm{CA}^0: \qquad \exists f^1 \forall x^0 \big(f(x) =_0 0 \leftrightarrow A(x) \big),$$

for arbitrary formulas $A(x)$ of $\mathcal{L}(\text{WE-PA}^\omega)$.
We've discussed how bar recursive functionals solve the ND-translation of this axiom, in fact, they solve $\mathrm{AC}^{0,\rho}$ via the DNS (Double Negation Shift).

Arithmetical comprehension

Today we'll discuss the special case of arithmetical comprehension:

$$\mathsf{CA}^0_{\mathsf{ar}}: \qquad \exists f^1 \forall x^0 \big(f(x) =_0 0 \leftrightarrow A_{\mathsf{ar}}(x) \big),$$

where $A_{\mathsf{ar}}(x)$ is *arithmetical*, that is, contains only quantifiers for type 0.

Proposition (11.11)

$\mathsf{CA}^0_{\mathsf{ar}}$ *is equivalent over* $\mathsf{WE\text{-}PA}^\omega$ *to:*

$$\Pi^0_1\text{-}\mathsf{CA}: \qquad \forall f^{0(0,0)} \exists g^1 \forall x^0 \left(g(x) =_0 0 \leftrightarrow \forall y^0 \left(f(x,y) =_0 0 \right) \right).$$

Let's prove this. Π^0_1-CA is a special case of $\mathsf{CA}^0_{\mathsf{ar}}$, so let's go the other way.

Proof of proposition 11.11, I

In WE-PA$^\omega$ + Π^0_1-CA, we are given an arithmetical formula $A_{\mathrm{ar}}(x, \bar{a})$, possibly containing x^0 free, but not f^1, with additional parameters $\bar{a}$. Our goal is:

$$\frac{}{\exists f^1 \forall x^0 \big(f(x) =_0 0 \leftrightarrow A_{\mathrm{ar}}(x, \bar{a})\big)}$$

We proceed by induction on $A_{\mathrm{ar}}(x, \bar{a})$. The cases are

$$A_{\mathrm{ar}} \equiv s =_0 t, \qquad A \wedge B, \qquad A \vee B, \qquad A \to B,$$

$$\exists y^0 A, \qquad \forall y^0 A.$$

Proof of proposition 11.11, II

The case $s =_0 t$. The goal is:

$$\frac{}{\exists f^1 \forall x^0 \big(f(x) =_0 0 \leftrightarrow s =_0 t\big)}$$

We simply use lambda abstraction and take $f = \lambda x^0.(s - t)$.

Proof of proposition 11.11, III

The case $A \wedge B$. The goal is:

$$\frac{\exists g^1 \forall x^0 \big(g(x) =_0 0 \leftrightarrow A\big) \qquad \exists h^1 \forall x^0 \big(h(x) =_0 0 \leftrightarrow B\big)}{\exists f^1 \forall x^0 \big(f(x) =_0 0 \leftrightarrow A \wedge B\big)}$$

Here we can take $f = \lambda x^0 . g(x) + h(x)$.
The cases $A \vee B$ and $A \rightarrow B$ are very similar.

Proof of proposition 11.11, IV

The case $\forall y^0 A$. The goal is:

$$\frac{\exists g^1 \forall x^0 \big(g(x) =_0 0 \leftrightarrow A\big)}{\exists h^1 \forall x^0 \big(h(x) =_0 0 \leftrightarrow \forall y^0 A\big)}$$

Let $f^{0(0,0)} = \lambda x, y . g(x)$. Feed f to Π^0_1-CA to get h^1 such that $h(x) = 0 \leftrightarrow \forall y(g(x) = 0)$, as desired.

The case $\exists y^0 A$ is very similar. □

Proposition 11.12

Arithmetical comprehension is implied by

$$\Pi^0_1\text{-AC:} \qquad \forall f^{0(0,0,0)}\big(\forall x \exists y \forall z\, f(x,y,z) = 0 \to \exists g^1 \forall x, z\, f(x, g(x), z) = 0\big).$$

Π^0_1-AC also implies arithmetical number choice:

$$\mathrm{AC}^{0,0}_{\mathrm{ar}}\text{:} \qquad \forall x^0 \exists y^0\, A_{\mathrm{ar}}(x,y) \to \exists f^1 \forall x^0\, A_{\mathrm{ar}}(x, f(x)).$$

Bar recursion of type 0,1

Consider the bar recursor of type $0,1$, $B_{0,1}$. It operates on sequence x^1, decider y^2, and is parameterized further by z of type $1(0,1)$ and u of type $\sigma = 1(1(0),0,1)$. So $B_{0,1}$ has type $1(2,1(0,1),\sigma,0,1)$ with defining axioms $BR_{0,1}$:

$$\begin{cases} y(\overline{x,n}) <_0 n \to B_{0,1}yzunx =_1 zn(\bar{x},n) \\ y(\overline{x,n}) \geq_0 n \to B_{0,1}yzunx =_1 u(\lambda d^0.B_{0,1}yzun'(\overline{x,n} * d))n(\bar{x},n) \end{cases}$$

Theorem 11.13

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Let $A(\underline{a})$ be an arbitrary formula of $\mathcal{L}(\text{WE-PA}^\omega)$. If

$$\text{WE-PA}^\omega + \text{QF-AC} + \text{AC}^{0,0}_{\text{ar}} \vdash A(\underline{a}),$$

then there is a tuple $\underline{t}$ of closed terms of $\text{WE-HA}^\omega + \text{BR}_{0,1}$, such that

$$\text{WE-HA}^\omega + \text{BR}_{0,1} \vdash \forall \underline{y} (A')_D (\underline{t}\,\underline{a}, \underline{y}, \underline{a}).$$

The verification can be carried out in qf-$(\text{WE-HA}^\omega) + (\text{BR}_{0,1})$.

Theorem 11.14

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Let $A(\underline{a})$ be an arbitrary formula of $\mathcal{L}(\widehat{\text{WE-PA}}^{\omega}\!\restriction)$. If

$$\widehat{\text{WE-PA}}^{\omega}\!\restriction + \text{QF-AC} + \text{AC}^{0,0}_{\text{ar}} \vdash A(\underline{a}),$$

then there is a tuple $\underline{t}$ of closed terms of $\widehat{\text{WE-HA}}^{\omega}\!\restriction + \text{BR}_{0,1}$ such that

$$\widehat{\text{WE-HA}}^{\omega}\!\restriction + \text{BR}_{0,1} \vdash \forall \underline{y}(A')_D(\underline{t}\,\underline{a}, \underline{y}, \underline{a}).$$

The verification can be carried out in qf-$(\widehat{\text{WE-HA}}^{\omega}\!\restriction) + (\text{BR}_{0,1})$.

Proposition 11.15

Proposition (11.15)

Let A be a prenex sentence of PA. *If* $\mathrm{PA} \vdash A$, *then there are closed terms* $\underline{\Phi}$ *of* $\widehat{\text{WE-HA}}^{\omega}\!\restriction + \mathrm{BR}_{0,1}$ *such that*

$$\widehat{\text{WE-HA}}^{\omega}\!\restriction + \mathrm{BR}_{0,1} \vdash \underline{\Phi}\ \textit{n.c.i.}\ A.$$

Compare with:

Proposition (10.9)

Let A be a prenex sentence of PA. *If* $\mathrm{PA} \vdash A$, *then there are closed terms* $\underline{\Phi}$ *of* WE-HA^{ω} *such that*

$$\text{WE-HA}^{\omega} \vdash \underline{\Phi}\ \textit{n.c.i.}\ A.$$