What Rests on What, Homotopically?

And Why a Little Bit Goes a Long Way

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"The mathematical sciences particularly exhibit order, symmetry, and limitations; and these are the greatest forms of the beautiful."
– Aristotle, *Metaphysics* M 3, $1078^{a36\text{–}b1}$

Outline

What Rests on What?

Feferman 1993

"Whenever a subject is organized systematically for expository or foundational purposes (or both), one must deal with the question: What rests on what? The way in which this is answered in the case of mathematics depends on whether one is considering it informally or formally, i.e. from the point of view of the mathematician or the logician, respectively. The latter usually deals with the question in terms of what specifically follows from what in a given logical/axiomatic setup. Proof theory provides technical notions and results which—when successful—serve to give a more global kind of answer to this question, in terms of reduction of one such system to another; moreover, these results provide a technical bridge from mathematics to philosophy."

Mathematics, Formal Systems, and Frameworks

Some logical notions of "resting on":

- $\mathcal{M}$ *rests on* T, in the sense that $\mathcal{M}$ *can be formalized* in T;
- φ *rests on* T, in the sense that φ *is provable in* T;
- T *rests on* $\mathcal{F}$, in the sense that T *is justified by* $\mathcal{F}$; and
- T_1 *rests on* T_2, in the sense that T_1 *is reducible to* T_2.

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Distinguish *conceptual* from *foundational* reductions:

Foundational Reduction (Relativized Hilbert's Program)

"A body of mathematics $\mathcal{M}$ is represented directly in a formal system T_1 which is justified by a foundational framework $\mathcal{F}_1$. T_1 is reduced proof-theoretically to a system T_2 which is justified by another, more elementary such framework $\mathcal{F}_2$."

We can add:

- $\mathcal{A}$ *rests on* $\mathcal{M}$, in the sense that $\mathcal{A}$ *applies* (or *is modeled via*) $\mathcal{M}$.

Example Frameworks and Reductions

The frameworks $\mathcal{F}$ might be some combination of:

- ultra-finitary, finitary, countably infinitary, uncountably infinitary.
- constructive or non-constructive,
- predicative or impredicative,

Some example reductions:

- (Gödel 1933) $\mathrm{PA} \leq \mathrm{HA}$ (non-constructive to constructive)
- (Friedman 1977, Sieg 1985) $\mathrm{RCA} + \mathrm{WKL} + \Sigma^0_1\text{-}\mathrm{IND} \leq_{\Pi^0_2} \mathrm{PRA}$ (countably infinitary to finitary)
- (Barwise–Schlipf 1975, Friedman 1976, Feferman–Sieg 1981) $(\Delta^1_1\text{-}\mathrm{CA})_0 \leq_{\Pi^1_0} \mathrm{PA}$ (uncountably infinitary to countably infinitary)
- (Friedman 1970, Feferman–Sieg 1981) $\Delta^1_1\text{-}\mathrm{CA} \leq_{\Pi^1_2} \mathrm{ACA}_{<\varepsilon_0}$ (impredicative to predicative)

Philosophical Import

Philosophical Import (*ibid.*)

"In general, the kinds of results presented here serve to sharpen what is to be said in favor of, or in opposition to, the various philosophies of mathematics such as finitism, predicativism, constructivism and set-theoretical realism. Whether or not one takes one or another of these philosophies seriously for ontological and/or epistemological reasons, it is important to know which parts of mathematics are in the end justifiable on the basis of the respective philosophies and which are not."

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Practical import: Mechanization of mathematics in proof-assistants provides further evidence for claims of the form $\mathcal{M}$ rests on T, while the logical analyses may inform the design of formal systems T that are suitable targets for mechanization.

Complementary program: Friedman–Simpson reverse mathematics.

Predicativity: Why a Little Bit Goes a Long Way

Feferman 1993

If one accepts the [Quine–Putnam] indispensability arguments, there still remain two critical questions:

(Q1) Just which mathematical entities are indispensable to current scientific theories?, and

(Q2) Just what principles concerning those entities are needed for the required mathematics?

Feferman then proceeds define a formal system W in which we can freely speak about real numbers, real functions, etc. However, the existence principles are weakened to allow a proof-theoretic reduction to PA.

Outline

The Homotopification of Mathematics

Manin 2009

I am pretty strongly convinced that there is an ongoing reversal in the collective consciousness of mathematicians: the right hemispherical and homotopical picture of the world becomes the basic intuition, and if you want to get a discrete set, then you pass to the set of connected components of a space defined only up to homotopy.

An implication of this is that for a type of mathematical objects A, the notion of identity $a =_A b$ between elements of A is no longer in general a simple truth-value (or proposition), but rather a homotopy type consisting of ways of identifying a with b.

How Did We Get Here?

- Pre-history: Abstraction and Axiomatization (E. Noether et al. 1920s)
- Homology is homotopy invariant, cohomology (1930s)
- Categories (Eilenberg–Mac Lane 1945)
- Weil conjectures & “Weil cohomology theory” (Weil 1949)
- Equivalences of categories (Grothendieck 1957)
- Cohomology theories spread all over mathematics (1960s–)
- Homotopical algebra, model categories (Quillen 1967)
- Quasi-categories (Boardman–Vogt 1973) [same year as Martin-Löf identity types]
- More general homotopy theories (Dwyer–Kan 1980s)
- The homotopy hypothesis and non-abelian cohomology (Grothendieck 1983, *Pursuing Stacks*)

Waldhausen

Algebraic K-theory of Spaces (Waldhausen 1983)

The first part of the paper, on which everything else depends, may perhaps look a little frightening because of the abstract language that it uses throughout. This is unfortunate, *but there is no way out*. It is not the purpose of the abstract language to strive for great generality. The purpose is rather to simplify proofs, and indeed to make some proofs understandable at all.

(Emphasis mine) Notice the significance for our philosophical questions!

How Did We Get Here? – continued

- Motivic cohomology settles the Milnor conjecture (Voevodsky 1993)
- Brave New Algebra (Elmendorf–Kriz–Mandell–May 1997, . . .)
- Higher Topos Theory (Lurie 2006)
- Growing realizations that the two main axiomatizations of quantum field theory both should be formulated ∞-categorically:
  - Functorial (Schrödinger picture): cobordism hypothesis
  - Algebraic (Heisenberg picture): higher sheaves of observables

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Lurie at the 2010 ICM

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"Other theory" includes: Better understanding of cohomology theories, moduli problems, the cobordism hypothesis, etc.

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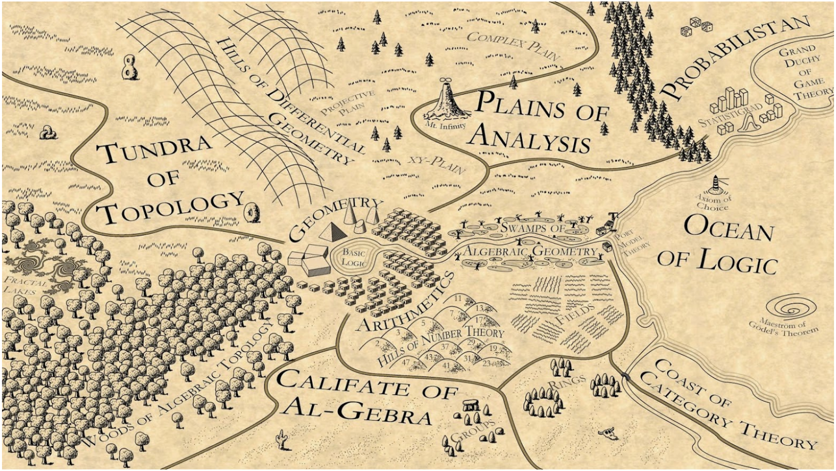
(I apologize for all the history and names I left out.)

Mathematatistan



(Image by Martin Kuppe 2014)

Outline

Homotopy Type Theory

We see the emergence not only of new bodies of homotopical mathematics, but also of

- a new range of *formal systems*: homotopy type theories: Martin-Löf type theory plus univalence plus . . .), and
- a new *foundational framework*: univalent (or structural) foundations, as opposed to non-univalent or concretistic foundations.

More precisely, we have a new distinction that we can apply independently of the old ones.

This gives many new instances of “what rests on what?”

Univalent foundations

The main new foundational requirement is that mathematical objects and concepts are represented by (homotopy) types in such a way that the identity types capture the mathematically natural notion of equivalence for these objects (structure identity principle). All constructions should respect equivalence.

It emphasizes the distinctions between

- Cauchy sequences and real numbers
- Extensional well-founded trees and (Cantorian) sets
- Strict (set-presented) categories and (univalent) categories
- Kan complexes and ∞-groupoids
- Quasi-categories and $(\infty, 1)$-categories
- etc.

It may further induce us to be agnostic whether certain abstract concepts have set presentations at all.

“It depends on what the meaning of ‘is’ is”

Some mathematicians hesitate to expand the use of the $=$ sign to encompass equivalence in general. Others take after Lady Welby:

Victoria Welby 1911 in *Significs and Language*

I am quite ready for the most drastic changes as well as for the most scrupulous and anxious preservation of our existing resources all over the world. [. . .] I want modes of expression as yet unused, though we must not say undreamt of, since there are many scientist’s and idealist’s diagrams, symbols and other ‘thinking machines’ all ready and in order, to rebuke us.

Formal Systems for HoTT

The basic formal system implementing univalent foundations consist of Martin-Löf's dependent type theory with Π,Σ,$\mathbb{N}$,0,$+$, and a hierarchy of *univalent* universes $\mathcal{U}$.

Additional principles include:

- Law of excluded middle (LEM), either
  - For all propositions
  - Just for small propositions
  - Just for propositions of the form $\exists_{(n:\mathbb{N})}(f\,n = 0)$ for all $f : \mathbb{N} \to 2$.
- All propositions are essentially small (resizing, impredicativity)
- Axiom of choice (AC)
- Countable choice (CC)
- Church's thesis: all functions $\mathbb{N} \to \mathbb{N}$ are computable
- Sets cover (SC)
- Whitehead's principle: homotopy groups detect equivalences
- The existence of various higher inductive types, such as pushouts

Global reductions for HoTT

Returning now to the question of relating univalent and non-univalent frameworks:

- Voevodsky's model of HoTT in simplicial sets provides a reduction of a classical univalent theory to a classical non-univalent theory.
- The cubical sets models of (Bezem–Coquand–Huber, Cohen–Coquand–Huber–Mörtberg, etc.) reduce a predicative univalent theory to a constructive, predicative non-univalent one. (The former is thus seen to be constructive as well.)

Although there are many further open questions regarding these models, they are in a sense in the "non-surprising" direction:

- Can we reduce a non-univalent theory to a univalent one while preserving as much set theory as possible?

Adequacy of HoTT

One of the major open problems in homotopy type theory is whether HoTT is adequate for actual homotopical mathematics:

- Can HoTT capture the theory of $(\infty, 1)$-categories?
- Can HoTT define the “homotopy type of” constructions on topological spaces, simplicial sets, etc.?
- Can HoTT formalize its own syntax and semantics, in particular in internal $(\infty, 1)$-toposes?

If HoTT by itself is not up to the task, we need new modes of expression.

One proposed extension is *two-level type theory*, but this is not justified by a univalent framework.

Local reductions for HoTT

Some of my recent work has revolved around the question whether bodies of mathematics that *prima facie* rest on non-univalent assumptions can be nevertheless be (indirectly) captured in a predicative, constructive, univalent system:

- The H-space structure on $\mathbb{S}^3$ (coming non-univalently from multiplication of unit quaternions) can be defined in HoTT with pushouts (B–Rijke)
- The homotopy types of the real and complex projective spaces can be constructed in HoTT with pushouts (B–Rijke)
- Cell complexes and cellular cohomology can be formalized in HoTT with pushouts (B–Favonia)
- The Leray spectral sequence of fibration can be formalized in HoTT with pushouts (van Doorn et al.)
- The homotopical circle as a HIT can be constructed in a basic HoTT with propositional truncation and no other HITs (Bezem–B–Grayson)

Outline

The periodic table

One particularly promising application of HoTT is to the study of higher groups: The collections of symmetries of general mathematical objects.

Definition

A k*-symmetric* n*-group* G is given by a pointed $(k-1)$-connected $(n+k-1)$-type $\mathrm{B}^k G$, the *classifying type* for G.
The type of *elements* of G is $\Omega^k \mathrm{B}^k G$.

$k \setminus n$	1	2	$\cdots$	∞
0	pointed set	pointed groupoid	$\cdots$	pointed ∞-groupoid
1	group	2-group	$\cdots$	∞-group
2	abelian group	braided 2-group	$\cdots$	braided ∞-group
3	— " —	symmetric 2-group	$\cdots$	sylleptic ∞-group
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
ω	— " —	— " —	$\cdots$	connective spectrum

Results on higher groups

Theorem (B–van Doorn–Rijke)

The stabilization and forgetting maps induce equivalences between

$$k\text{-symmetric } n\text{-groups}$$

and

$$(k+1)\text{-symmetric } n\text{-groups}$$

whenever $k \geq n+1$.

Theorem (B–Rijke)

For any fiber sequence $F \hookrightarrow E \twoheadrightarrow B$ *we obtain a long* n*-exact sequence*

$$\cdots \longrightarrow \pi_k^{(n)}(F) \longrightarrow \pi_k^{(n)}(E) \longrightarrow \pi_k^{(n)}(B) \longrightarrow \cdots$$

of homotopy n*-groups, where the morphisms are homomorphisms of* k*-symmetric* n*-groups whenever the codomain is a* k*-symmetric* n*-group.*

Cohomology and New Descriptions of Classifying Types

The cohomology set of X with coefficients in a group G is the set of connected components of the function type $X \to \mathrm{B}G$. Thus, in order to facilitate cohomology computations, it is advantageous to have as many descriptions of classifying types as possible.

Here are some results in that direction:

- For any ∞-group B, $\mathrm{B}G$ is the type of G-torsors (B)
- For any ∞-group G, $\mathrm{B}G$ is the colimit of generalized projective spaces $(G * \cdots * G) \,/\!\!/\, G$, viz. Milnor's *join construction* (B–Rijke)
- For any 1-group G, $\mathrm{B}^2\mathrm{Z}(G)$ is the type of G-banded gerbes (B)
- For any abelian 1-group A, $\mathrm{B}^k A$ is the type of A-banded $(k-1)$-gerbes (B)
- For any 2-group G presented by a crossed module $V \to H$, $\mathrm{B}G$ is the type of bitorsors for the crossed module, and the type of V-banded H-gerbes (B–Capriotti).

Outline

A Little Bit (Still) Goes a Long Way

Suppose *hypothetically* that string theory turns out to be the correct framework for quantum gravity.

A good case can be made that a *direct axiomatization* (cf. Hilbert's sixth problem) will be in terms of *cohesive homotopy type theory* (Schreiber, Shulman et al.)

This is a modal extension of HoTT with modalities:

- fermionic $\dashv$ bosonic $\dashv$ rheonomic
- reduced $\dashv$ infinitesimal $\dashv$ étale
- homotopic $\dashv$ discrete $\dashv$ codiscrete

The smooth string 2-group

The homotopy groups of the shape of BO are $0, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, \ldots$.

Since the shape of $\mathrm{U}(1)$ is $\mathrm{B}\mathbb{Z}$, the Whitehead tower of BO has a smooth refinement:

$$
\begin{array}{ccccccc}
\mathbb{B}\mathrm{String} & \longrightarrow & 1 & & & & \\
\downarrow & & \downarrow & & & & \\
\mathbb{B}\mathrm{Spin} & \longrightarrow & \mathbb{B}^3\mathrm{U}(1) & \longrightarrow & 1 & & \\
\downarrow & & \big| & & \downarrow & & \\
\mathbb{B}\mathrm{SO} & \longrightarrow & \big| & \longrightarrow & \mathrm{B}^2\mathbb{Z}/2 & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{B}\mathrm{O} & \longrightarrow & \|\mathrm{BO}\|_4 & \longrightarrow & \|\mathrm{BO}\|_2 & \longrightarrow & \|\mathrm{BO}\|_1 = \mathrm{B}\mathbb{Z}/2
\end{array}
$$

defining the smooth 2-group String.

Conclusion

The homotopification of mathematics brings in a new foundational distinction, leading to new formal systems, and many new philosophically relevant problems for proof theory broadly understood:

- Formulating adequate type theories justified by univalent frameworks.
- Extending realizability, forcing, and Dialectica interpretations to give new reductions.
- Designing practical proof assistants for univalent type theories, and investigating impacts for automation.
- Investigating in how far homotopical mathematics that originally relied on non-univalent principles can be formalized in univalent type theories.
- Investigating the range of simplifications achieved in applications.