From Explicit Mathematics to Synthetic Mathematics – and back?

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50 Years of Explicit Mathematics

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Outline

Explicit Mathematics for Constructive Mathematics

Original aim of EM: Logical framework for (Bishop-style) constructive mathematics:

> *5.1. Constructivity is understood here in the sense of intuitionism. Bishop [67], [70] takes a more restrictive position but within which he redevelops substantial portions of mathematics . . .*
>
> *(1) Claim.* T_0 *is constructively correct.* (Feferman 1975)

The claim is justified by interpreting operations as constructions (with $fx \simeq y$ when the construction f applied x yields y) and classes as (intensional) types.

No existence property, unless $\exists$ is banned from comprehension (T_0^-), and no disjunction property, unless $\vee$ is as well (T_0^{--}).

Since we're *explicit* about constructions, there's no harm in even adding LEM to T_0.

Implicit Foundations for Constructive Mathematics

The alternative foundations for constructive mathematics are *implicit* in the sense that intuitionistic logic implicitly guides construction. Leads to the disjunction and existence properties, or more generally *canonicity* for positive types.

Hilbert and Bernays (1934) stated that $\exists x.A(x)$ is an *incomplete* communication of a more involved statement $A(m)$ for an *explicit* m.

Examples:

- Heyting arithmetic in finite types HA^ω has EP.
- Myhill's CST^- has EP.
- Aczel's $\mathrm{CZF}_\mathcal{E}$ and its extension with power sets, $\mathrm{CZF}_\mathcal{P}$, have EP.
- Martin-Löf type theories generally have canonicity.
- Homotopy type theory has homotopy canonicity (cubical type theories have canonicity).
- IZF_R has EP.

EP for set theories is subtle; can fail for constructive theories and hold for classical ones (Swan 2014).

The axiom of unique choice

A key desideratum for implicit construction is the axiom of unique choice. For constructive set theories, it follows from replacement, and in (homotopy) type theory it follows from the definition of unique existence as contractibility.

In Sambin and Maietti's *minimalist foundations* and in the Calculus of (Inductive) Constructions, it fails to allow for explicit (& logic-enriched) interpretations.

Extensionality

More generally, the implicit approach facilitates abstraction (information hiding) via extensionality principles: Function extensionality, propositional extensionality, and ultimately type extensionality (univalence).

Closely related with the ability to form quotients: propositional truncation, and higher inductive types.

- $P \vee Q$ is the pushout of $P \leftarrow P \times Q \rightarrow Q$ – to use the assumption that $P \vee Q$ you give constructions assuming P and assuming Q, but they must agree when both hold.
- $\exists x : A.\ P(x)$ is the propositional truncation of $\Sigma x : A.\ P(x)$.

Extensional Mathematics in Explicit Mathematics

In EM, we can interpret extensional concepts à la Bishop as setoids. This also gives quotients. Jaun (2019) analyzed the various versions of Bishop sets, including fully explicit ones (the exact completion).

It still remains to interpret higher types as infinity groupoids in EM.

In any case, these interpretations validate the presentation axiom (every type is covered by a projective one), so EM is *not* suitable for reasoning extensionally in models where this fails.

Models of constructive mathematics

Two main classes of models of constructive mathematics:

- Topological (starting with Stone 1937 and Tarski 1938). Then via Kripke and Beth (and Cohen forcing) eventually culminates in Grothendieck–Lurie $(\infty, 1)$-topos semantics in (higher) sheaves $\mathrm{Sh}(C) \hookrightarrow \mathrm{PSh}(C) \equiv [C^{\mathrm{op}}, \mathcal{S}]$. (Toposes as the natural generalization of topological spaces.)
- Realizability (starting with Kleene 1941/5). Axiomatization of the Brouwer–Heyting–Kolmogorov proof interpretation, recognizing the key role of *partiality*. Any PCA gives rise to a *realizability topos* (an elementary topos in impredicative metatheories, or a $\Pi\mathrm{W}$-pretopos otherwise), and models of EM.

Other interpretations:

- Modal interpretations (starting with Gödel 1933): Not clear (to me) if much is gained over Kripke models?
- Dialectica/functional interpretations (starting with Gödel 1941/1958): Doesn't model extensionality.

Benefits of models of constructive mathematics

The benefit of these models – along with other proof interpretations! – is to give further answers to Kreisel's question:

> *What more do we know if we have proved a theorem by restricted means than if we merely know that it is true?*

For EM, we have models over PCAs, giving constructive, recursive, hyperarithmetic, and classical versions of each theorem. – It seems hard to give interesting sheaf models.

Interlude: Constructive reverse mathematics

What's a convenient base system for constructive reverse mathematics? (Diener 2018)

Some desiderata:

- Basic strength at PRA.
- Flexible enough to formalize finitist mathematics without too much coding.
- Flexible enough to state constructions and results of abstract mathematics.
- Compatible with extensions in proof theoretic strength up to classical set theory with power sets, large cardinals, etc.
- Suitable for investigations in classical reverse mathematics, assuming LPO.
- Fully extensional/implicit, but compatible with explicit principles (presentation axioms).
- ...

Made some progress with a primitive recursive dependent type theory (Buchholtz and Schipp von Branitz 2024), though not yet a primitive recursive HoTT.

Synthetic Mathematics

> *. . . large parts of modern mathematical research are based on a dexterous blending of axiomatic and constructive procedures.* (Weyl 1985)

Closely related to Feferman's idea of *local foundations*: What are the essential features of a realm of mathematics and how should they be axiomatized?

Primordial example is Euclidean geometry of basic figures, that are *postulated* rather than *analyzed* in terms of something else/more primitive, and whose basic properties are taken as *axioms*.

Another closely related idea: *purity of methods*

Russell said this has all the benefits of theft over honest toil! But the toil of constructing models repays what was stolen.

Modern examples of synthetic mathematics

- Synthetic differential geometry: The differential line R is postulated as $\mathbb{Q}$-algebra with Kock–Lawvere axiom
$$R[x]/(x^2) \xrightarrow{\sim} (\{\, d : R \mid d^2 = 0 \,\} \to R)$$
- Synthetic algebraic geometry: The algebraic line R is postulated as a local ring satisfying
$$A \xrightarrow{\sim} (\mathrm{Spec}(A) \to R),$$
where $\mathrm{Spec}(A) := \mathrm{Hom}_R(A, R)$, for all f.p. R-algebras A. Also: Zariski local choice.
- Synthetic computability theory: Assume Church's thesis
- Synthetic domain theory: Assume partial map classifier Σ, a bounded distributive lattice with Phoa's principle
$$(\Sigma \to \Sigma) \xrightarrow{\sim} \sum_{x,y:\Sigma} x \leq y$$

More examples of synthetic mathematics

- Synthetic (higher) category theory: Extends synthetic domain theory with Σ totally ordered, and the universe generated by the simplices $\Delta^n \subseteq \Sigma^n$. With Gratzer and Weinberger, I've developed a lot of higher category in this setting. (Gratzer, Weinberger, and Buchholtz 2024; Gratzer, Weinberger, and Buchholtz 2025)
- Synthetic Stone duality (Cherubini et al. 2024): Use the boolean algebra 2 with the axiom:
  - Stone duality:
  $$B \xrightarrow{\sim} (\mathrm{Spec}(B) \to 2)$$
  for all countably presented boolean algebras B, where $\mathrm{Spec}(B) := \mathrm{Hom}(B, 2)$.
  - An algebra homomorphism $B \to C$ is injective if and only if $\mathrm{Spec}(C) \to \mathrm{Spec}(B)$ is surjective.
  - Local choice with respect to surjections $\mathrm{Spec}(C) \to \mathrm{Spec}(B)$.
  - Dependent choice.

  This implies Markov's principle and LLPO and facilitates the study of Stone spaces and their quotients, e.g., every function $[0, 1] \to [0, 1]$ is ε–δ-continuous.

Synthetic Mathematics eats itself

One of my current projects is to develop the *synthetic mathematics of synthetic mathematics*: Postulate constructions of theories, and give axioms that allow us to derive the "local" axioms for each theory. (Joint with Mark Williams and Johannes Schipp von Branitz)

Intended model is built on the multiverse of toposes (Blechschmidt and Oldenziel 2023) (extending the set-theoretic multiverse).

Blechschmidt (2023) gave a rule for deriving the local axioms, via *synthetic quasicoherence* (a fully generalized *Nullstellensatz*).

David Jaz Myers analyzed this as an analog of Martin-Löf's induction principle paths:

$$\begin{tikzcd} E \arrow[r, "x"] \arrow[d, "i"'] & X \arrow[d, "\mathrm{qcoh}"] \\ E^{\Sigma} \arrow[r, "\mathrm{id}"] \arrow[ur, dashed] & E^{\Sigma} \end{tikzcd}$$

Synthetic Topos Theory

We postulate that Σ is a $\mathcal{U}_0$-small frame satisfying a duality axiom. Ongoing work: Deriving the synthetic quasicoherence principle.

Next step will be to replace Σ with an object classifier.

Obstacle: This only captures small maps between small theories. For example: Expanding the universe give new models of a theory (relative to any other theory).

Poincaré predicativity

> *In the words of Poincaré, the definitions used of objects in an incomplete totality should not be "disturbed by the introduction of new elements."* *(Feferman 2002)*

- The universe of operations in EM is open.
- The collection of types (and hence of constructions) in MLTT is open.

Questions:

- To what extent are function types predicative under these interpretations? (Cf. the blowup in strength for MLTT+LEM.)
- Can an analysis via EM help with the predicativity problems of synthetic theory of theories?

Thank you

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