Genuine pairs and the trouble with triples in homotopy type theory

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What is an unordered pair in type theory?

After having studied equality and singletons in HoTT, it's time to tackle unordered pairs!

In set theory, unordered pairs are a basic notion (even the subject of an axiom), while in type theory, they're not basic at all. We can however ask:

What is an unordered pair of groups? Of categories? Of elements of a type X?

Two reasonable answers:

- The set quotient $\mathrm{sUP}(X)$ of $X \times X$ modulo the equivalence relation generated by $(x_0, x_1) \sim (x_1, x_0)$.
- The type $\mathrm{hUP}(X) :\equiv (K : \mathrm{B}\Sigma_2) \times (\mathrm{El}\, K \to X)$, where $\mathrm{B}\Sigma_2 :\equiv (K : \mathcal{U}) \times \|K \simeq 2\|$ and El is the first projection.

Note that $\mathrm{sUP}(\mathbb{S}^1) \simeq 1$ and $\mathrm{hUP}(1) \simeq \mathrm{B}\Sigma_2$ is the (homotopy type) of infinite dimensional real projective space, $\mathbb{RP}^\infty$.

After recalling some background on HoTT, I'll give a third answer (also proposed by Paolo Capriotti), prove a theorem about it, and then say something about triples and finite multisets.

Outline

Questions welcome!

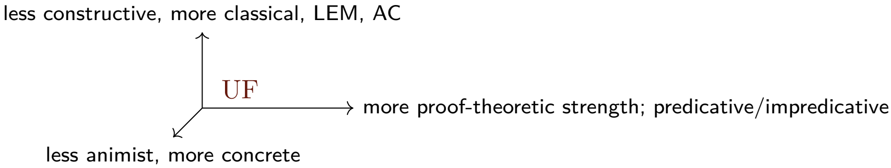
Homotopy Type Theory and Univalent Foundations

groupoids will do for this talk

- Basic idea: types are homotopy types (∞-groupoids, aka animas), elements of the identity type $a =_A b$ are *identifications* (or *paths*) in A.
- Formal system: Martin-Löf type theory with Voevodsky's *univalence axiom*,

  the map $\mathrm{idtoequiv} : A =_{\mathcal{U}} B \to A \simeq B$ is an equivalence,

  extended with *higher inductive types* HITs, such as pushouts.
- Univalent Foundations: this works as a foundational system compatible with many different philosophical commitments:



- Also: Internal language of $(\infty, 1)$-toposes (domain specific languages).

Higher inductive types: pushouts

The *pushout* of a span $A \xleftarrow{f} C \xrightarrow{g} B$ is the higher inductive type D defined by two maps $\mathrm{inl} : A \to D$ and $\mathrm{inr} : B \to D$, and a dependent function $\mathrm{glue} : (x : C) \to \mathrm{inl}(f(x)) = \mathrm{inr}(g(x))$, making a square

D

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\Nearrow\mathrm{glue}"] & B \arrow[d, "\mathrm{inr}"] \\ A \arrow[r, "\mathrm{inl}"'] \arrow[ur, phantom, "\ulcorner", very near start] & D, \end{tikzcd}$$

satisfying the universal property (equivalently: the induction principle).

From pushouts we can construct the circle, suspensions (spheres), general finite colimits, n-truncations (using Rijke's join construction), and we get the replacement principle (the image of a map from a small type to a locally small type is essentially small).

$\mathcal{U} \hookrightarrow \mathcal{V}$

Recent progress: meta-theory

Lots of recent progress on meta-theory of HoTT:

- Shulman: Any Grothendieck $(\infty, 1)$-topos can be presented by a model of HoTT (arXiv:1904.07004).
- Kapulkin–Sattler: Homotopy canonicity for HoTT (2019).
- Awodey–Cavallo–Coquand–Riehl–Sattler: Constructive model of HoTT that classically presents ∞-groupoids (2019).
- Sterling–Angiuli: Normalization for cubical type theory (arxiv:2101.11479)

Recent progress: theory

And on internal theory:

- Ahrens–North–Shulman–Tsementzis: Univalence Principle (arXiv:2102.06275)
- Anel–Biedermann–Finster–Joyal: Generating lex modalities (arXiv:2101.02791)
- B–Rijke: Higher long exact sequences of a fibration (arXiv:1912.08696)
- Christensen–Rijke: Characterizations of modalities (arXiv:2008.03538)
- Christensen–Scoccola: The Hurewicz theorem (arXiv:2007.05833)
- Myers: Modal fibrations (arXiv:1908.08034), & Higher Schreier theory (2020)
- Swan: Nielsen–Shreier theory (arXiv:2010.01187)
- Rijke–Cherubini: Modal descent (arXiv:2003.09713)

And on using extra structure in particular $(\infty, 1)$-toposes:

- Riley–Finster–Licata: Synthetic Spectra (arXiv:2102.04099)
- B–Weinberger: (Co)cartesian fibrations of synthetic $(\infty, 1)$-categories (arXiv:2105.01724)

Exciting talks at this meeting too!

(Sorry for leaving some out!)

Long-standing open problems

Open problem 1

Define $(\infty, 1)$-categories and develop their theory.
One way: Define the type of semi-simplicial types.

This is the main stumbling block towards more wide-spread adoption of HoTT.

Two-level type theory is a principled work-around with other possible applications.

Open problem 2

Show that the suspension of a (pointed) set is a groupoid.

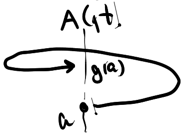
The theorem I'll show later makes me slightly more hopeful that these have positive solutions.

Outline

Group theory from the HoTT point of view

Types are ∞-groupoids $\therefore$ Pointed, connected types BG are ∞-groups (with type of elements $G :\equiv \Omega BG$, identity element the reflexivity path and composition of paths as operation)

$= (pt =_{BG} pt)$



A type with a G-action is a type family $A : BG \to \mathcal{U}$.

We have the (wild) adjoint triple

$$
\begin{tikzcd}
(BG \to \mathcal{U}) \arrow[r, shift left=4, "(-)_{hG}"] \arrow[r, shift right=4, "(-)^{hG}"'] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathcal{U} \arrow[l, "\Delta" description]
\end{tikzcd}
$$

where $A_{hG} :\equiv (t : BG) \times A(t)$ is the homotopy orbit type, and $A^{hG} :\equiv (t : BG) \to A(T)$ is the homotopy fixed point type.

(For more, see the paper (with van Doorn and Rijke) *Higher Groups in Homotopy Type Theory* and two manuscripts in preparation: *The Symmetry Book*, and *Higher Group Theory in Homotopy Type Theory*)

Homotopy unordered pairs

$B\Sigma_2 = (K : \mathcal{U}) \times \| K = 2 \|$

$\simeq \mathrm{HIT}\big(\mathrm{pt} : B\Sigma_2,$
$g : \mathrm{pt} = \mathrm{pt},$
$! : g^2 = \mathrm{refl},$
$! : \mathrm{isGroupoid}(B\Sigma_2)\big)$

We have a Σ_2-action on $X \times X$ given by

$$\mathrm{hEP}(X) :\equiv K \mapsto (\mathrm{El}\, K \to X) : \mathrm{B}\Sigma_2 \to \mathcal{U}$$

since $(2 \to X) \simeq X \times X$. The non-trivial element of Σ_2 maps a pair (x_0, x_1) to (x_1, x_2).
Then $\mathrm{hEP}(X)_{\mathrm{h}\Sigma_2} = \mathrm{hUP}(X)$ and $\mathrm{hEP}(X)^{\mathrm{h}\Sigma_2} = X$. b/c $(K : B\Sigma_2) \times \mathrm{El}\, K$ is contractible

Note that $\|\mathrm{hUP}(X)\|_0 \simeq \mathrm{sUP}(X) \simeq (P : X \to \mathrm{Prop}) \times \exists_{x_0, x_1 : X}(P = \{x_0, x_1\})$.
$(\lambda y .\ y = x_0 \vee y = x_1)$

For modeling coherently commutative binary operations, $\mathrm{hUP}(X)$ is what we want.

For modeling (non-planar) binary trees, maybe it's not right?

For some applications we want a notion of unordered pairs in X, $\mathrm{UP}(X)$, such that $\mathrm{UP}(X)$ is a set whenever X is.
NB $\mathrm{hUP}(X)$ doesn't satisfy this, as $\mathrm{hUP}(1) = \mathrm{B}\Sigma_2$.

Borrowing from equivariant homotopy theory

In the equivariant homotopy theory of a finite group G, we study the $(\infty, 1)$-category presented by G-spaces where we invert equivariant maps that induce weak equivalences on all fixed point spaces for all subgroups of G.

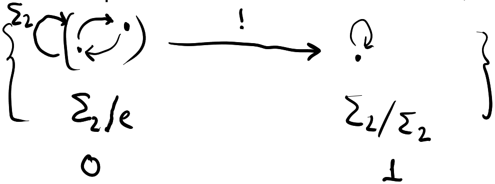
(Generalized) Elmendorf's theorem: The resulting $(\infty, 1)$-category is an $(\infty, 1)$-topos equivalent to presheaves on the orbit category of G.

Definition

The orbit category of a group G, Orb_G, is the full subcategory of the category of G-sets on the transitive G-sets.

Each subgroup (up to conjugation) H corresponds to such a transitive G-set, namely the orbit space G/H (hence the name!).

For $G = \Sigma_2$, we get:



Presheaves on direct categories

Shulman: *Univalence for inverse EI diagrams* (arXiv:1508.02410) showed that we can define diagrams on finite inverse EI categories, hence presheaves on direct EI categories in HoTT. (And this leads to models of HoTT.) — Here the example of presheaves on Orb_G for G a compact Lie group was also highlighted.

Fact: For every compact Lie group G, the orbit category Orb_G (here corresponding to closed subgroups) is a direct EI category.

$\left(0 \quad 1 \quad \overset{\text{terminal}}{2} \right) = I^{\triangleright}$

Example application: In presheaves on $I :\equiv \{* \xrightarrow{\mathbb{S}^1} *\}$, there is no set covering Set. In presheaves on $I^{\triangleright}$, the proposition $\exists_{X:\mathrm{Set}}$(surjection X to Set) is false.

NB The semi-simplex category Δ_{inj} is a direct category.

The orbit category of Σ_2

$$\begin{aligned}
\mathrm{Orb}^{\mathrm{core}}_{\Sigma_2} &:= (X : \mathrm{B}\Sigma_2 \to \mathrm{Set}) \times \mathrm{isConn}_0((J : \mathrm{B}\Sigma_2) \times X\,J) \times \mathrm{hasDecEq}(X\,\mathrm{pt}) \\
&= (X : \mathrm{B}\Sigma_2 \to \mathcal{U}) \times \Big(\|X = \mathrm{El}\| + \|X = \mathrm{Triv}\|\Big) \\
&= \mathrm{B}\Sigma_2 + 1,
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{El}(J) &= J = (\text{the type of elements of the 2-element set } J) \\
\mathrm{Triv}(J) &= 1 = (\text{the unit type}).
\end{aligned}$$

A genuine Σ_2-type A now consists of

$$\begin{aligned}
A_0 &: \mathrm{B}\Sigma_2 \to \mathcal{U} \\
A_1 &: A^{\mathrm{h}\Sigma_2} \to \mathcal{U}
\end{aligned}$$

Here, A_0 is the underlying homotopy Σ_2-action, while A_1 is the data of when a homotopy Σ_2-fixed point is genuine.

A homotopy colimit calculation

We have an adjoint quadruple,

$$\begin{tikzcd} \mathrm{PSh}(\mathrm{Orb}_{\Sigma_2}) \arrow[r, shift left=3, "(-)_{\Sigma_2}"] \arrow[r, shift right=1, "(-)^{\Sigma_2}"'] & \mathcal{U} \arrow[l, shift right=1, "\Delta"'] \arrow[l, shift left=3, "\nabla"] \end{tikzcd}$$

$\nabla A_0 K = 1$ $\qquad$ $\Delta A_0 K = A$

$\nabla A_1 v = A$ $\qquad$ $\Delta A_1 v = 1$

$(A)^{\Sigma_2} = (v : A^{h\Sigma_2}) \times A_1 v$

Theorem

The colimit of A, can be expressed as the following pushout

$$\begin{tikzcd} \mathrm{B}\Sigma_2 \times A^{\Sigma_2} \arrow[r, "\mathrm{pr}_2"] \arrow[d] & A^{\Sigma_2} \arrow[d, "\text{points-to-pieces}"] \\ A_{\mathrm{h}\Sigma_2} \arrow[r, "q"'] & A_{\Sigma_2} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

where the left map takes $(K, (v, w))$ to $(K, v\,K)$.

Genuine pairs

For pairs, every homotopy fixed point of $\mathrm{hEP}(X)$ is genuine, since $(K : \mathrm{B}\Sigma_2) \to (\mathrm{El}\, K \to X) \simeq X$. This means that $\mathrm{EP}(X)$, the type of equivariant pairs in X has:

$$\begin{aligned} \mathrm{EP}(X)_0\, J &= \mathrm{hEP}(X) = (\mathrm{El}\, J \to X) \\ \mathrm{EP}(X)_1\, v &= 1 \end{aligned}$$

$(X \times X)^{\Sigma_2} = X$

By the colimit calculation, the categorical orbit type $\mathrm{UP}(X)$ is the pushout:

$$\begin{tikzcd} (K, x) \arrow[d, mapsto] & \mathrm{B}\Sigma_2 \times X \arrow[r, "\mathrm{pr}_2"] \arrow[d] & X \arrow[d, "s"] \\ (K, \mathrm{cst}_x) & \mathrm{hUP}(X) \arrow[r, two heads, "q"'] & \mathrm{UP}(X) \arrow[ul, phantom, "\ulcorner", very near start] \\ & X \times X \arrow[u] & \end{tikzcd}$$

Some ingredients

Theorem (Fundamental theorem of identity types)

Let A be a type, $a : A$, $B : A \to \mathcal{U}$, and $b : B(a)$. If $(x : A) \times B(x)$ is contractible, then the induced maps $(a =_A x) \to B(x)$ are equivalences.

Definition

The join $A * B$ of two types A and B is the pushout

$$\begin{tikzcd}
A \times B \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & B \arrow[d, "\mathrm{inr}"] \arrow[ddr, bend left, "r"] & \\
A \arrow[r, "\mathrm{inl}"'] \arrow[drr, bend right, "l"'] & A * B \arrow[ul, phantom, "\ulcorner", very near start] & \\
& & X
\end{tikzcd}$$

(higher version of disjunction!)

Lemma

Let P be a proposition. If A is an n-type, then so is $A * P$.

A theorem

Theorem (Conjectured by Capriotti)

If X is a set, then so is $\mathrm{UP}(X)$.

To prove this we fix $(x_0, x_1) : X^2$, let $P :\equiv (x_0 = x_1)$, and consider the type family $Q : \mathrm{UP}(X) \to \mathcal{U}$ defined by

$$
\begin{aligned}
Q(q(K,w)) &= ((K,w) =_{\text{hUP}} (2,(x_0,x_1))) * (P \times (w = \mathrm{cst}_{x_0})) \\
Q(s(x)) &= (x = x_0) \times (x = x_1) \\
[Q](\mathrm{glue}(K,p)) &= \text{“true is true”}
\end{aligned}
$$

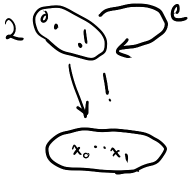
Certainly, Q is valued in sets. If we show that $Q(z) \simeq (q(x_0,x_1) =_{\mathrm{UP}(X)} z)$, then we see that Q is actually valued in propositions:

Suffices to check $Q(q(2,(x_0,x_1))) = ((2,(x_0,x_1)) = (2,(x_0,x_1))) * P$

inr p ✓

inl $(e : 2 = 2, !)$: either $e = \mathrm{id}_2$ ✓

or $e = \mathrm{swap}_2$, in which case $x_0 = x_1$ $\therefore P$ ✓



Proof sketch

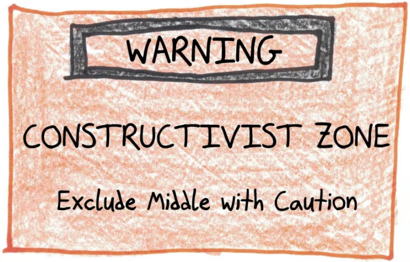
It thus suffices to show that $D :\equiv (z : \mathrm{UP}(X)) \times Q(z)$ is contractible. By the flattening lemma, D expressible as a pushout

$$\begin{tikzcd}
\overset{0}{\mathrm{B}\Sigma_2 \times \cancel{P}} \arrow[r, "\mathrm{pr}_2"] \arrow[d] & \overset{0\quad 1}{P} \arrow[d, "\mathrm{inr}"] \\
\mathrm{B}\Sigma_2\ \ \underset{1}{R} \arrow[r, "\mathrm{inl}"'] & \underset{1}{D}\ \ 1 \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
\quad \text{where} \quad
\begin{tikzcd}
0\ \ \overset{1}{P} \arrow[r] \arrow[d] & \overset{\mathrm{B}\Sigma_2}{\mathrm{B}\Sigma_2 \times P}\ \ 0 \arrow[d, "\mathrm{inr}"] \\
1 \arrow[r, "\mathrm{inl}"'] & \underset{1}{R}\ \ \mathrm{B}\Sigma_2 \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

$P = 1$

$P = 0$

Now it's easy if LEM holds:



credit: Ben Orlin, mathwithbaddrawings.com

Proof sketch, continued

(skipped for lack of time, see)

Mechanization of main step: ✓✓

https://github.com/UlrikBuchholtz/cubical/blob/unordered-pairs/Cubical/Experiments/Pairs.agda

Outline

The orbit category of Σ_3

The groupoid of objects of the orbit category Orb_{Σ_3} is

$$\begin{aligned}\mathrm{Orb}^{\mathrm{core}}_{\Sigma_3} &:= (X : \mathrm{B}\Sigma_3 \to \mathrm{Set}) \times \mathrm{isConn}_0((J : \mathrm{B}\Sigma_3) \times X\,J) \times \mathrm{hasDecEq}(X\,\mathrm{pt}) \\ &= (X : \mathrm{B}\Sigma_3 \to \mathcal{U}) \times \Big(\|X = \mathrm{Ord}\| + \|X = \mathrm{El}\| + \|X = \mathrm{Cyc}\| + \|X = \mathrm{Triv}\|\Big) \\ &= \mathrm{B}\Sigma_3 + (1 + \mathrm{B}\Sigma_2) + 1,\end{aligned}$$

where

$$\begin{aligned}\mathrm{Ord}(J) &= (\mathrm{Fin}\,3 = J) = (\text{the type of total orders on } J) \\ \mathrm{El}(J) &= J = (\text{the type of elements of the 3-element set } J) \\ \mathrm{Cyc}(J) &= (f : J \to J) \times \|(J, f) = (\mathrm{Fin}\,3, \mathrm{succ})\| = (\text{the type of cyclic orders on } J) \\ \mathrm{Triv}(J) &= 1 = (\text{the unit type}).\end{aligned}$$

These correspond to subgroup-up-to-conjugation $i_H : BH \to \mathrm{B}\Sigma_3$.
Note that Σ_2 acts on the C_3-subgroups: For every $K : \mathrm{B}\Sigma_2$, we have the orbit $X_K\,J = (K = \mathrm{Cyc}(J))$ with corresponding subgroup-up-to-conjugation $i_K : \mathrm{BC}_3^{(K)} \to \mathrm{B}\Sigma_3$, meaning Σ_2 acts on the homotopy C_3-fixed points of A.

Genuine Σ_3-types

A genuine Σ_3-type A now consists of

$$A_0 : \mathrm{B}\Sigma_3 \to \mathcal{U}$$
$$A_2 : A^{\mathrm{h}\Sigma_2} \to \mathcal{U}$$
$$A_3 : (K : \mathrm{B}\Sigma_2) \to A^{\mathrm{hC}_3^{(K)}} \to \mathcal{U}$$
$$A_4 : (v : A^{\mathrm{h}\Sigma_3}) \to (u : A_2\,(i^*_{\Sigma_2}\,v)) \to (t : (K : \mathrm{B}\Sigma_2) \to A_3\,(K, i^*_{\mathrm{C}_3^{(K)}}\,v)) \to \mathcal{U}$$

Here, A_0 is the underlying homotopy Σ_3-action, while A_2 is the data of when a homotopy Σ_2-fixed point is genuine.
Then A_3 is the data of when any point in the Σ_2-orbit type of homotopy C_3-fixed points is a genuine C_3-fixed point.
Finally, A_4 is the data of when a homotopy Σ_3-fixed point is genuine, indexed by the data that the induced Σ_2- and $\mathrm{C}_3^{(K)}$-fixed points are all genuine.

Genuine unordered triples

$\Sigma_3 \curvearrowright X \times X \times X$

The genuine equivariant type of triples in X is

$$\begin{aligned}
\mathrm{ET}(X)_0\, J &= (\mathrm{El}\, J \to X) \\
\mathrm{ET}(X)_2\, v &= 1 \\
\mathrm{ET}(X)_3\, (K, v) &= 1 \\
\mathrm{ET}(X)_4\, v\, u\, t &= (x : X) \times (v =_{(\mathrm{El}\, J : \mathrm{B}\Sigma_3) \to \mathrm{El}\, J \to X} \mathrm{cst}_x)
\end{aligned}$$

We can write down the colimit of a presheaf over Orb_{Σ_3} either as a HIT or in terms of pushouts. (I haven't done this yet!) This will define the type of genuine unordered triples in a type.

Conjecture

The type of unordered triples in a set X is again a set.

Outline

Genuine finite multisets

It seems unfeasible to deal with general genuine finite multisets in the same way, simply because the system of subgroups of Σ_n becomes increasingly unwieldy as n increases.

If we could, we'd like to set

$$\mathrm{MS}(X) :\equiv (n : \mathbb{N}) \times (X^n)_{\Sigma_n}$$

In comparison, we have the homotopy finite multisets

$$\begin{aligned} \mathrm{hMS}(X) &:\equiv (K : \mathrm{FinSet}) \times (\mathrm{El}\, K \to X) \\ &\simeq (n : \mathbb{N}) \times (X^n)_{\mathrm{h}\Sigma_n} \end{aligned}$$

Outlook / lessons?

- We have defined the types of genuine unordered pairs and indicated the troubles with ascending to triples and larger finite multisets.
- The type $\mathrm{UP}(X)$ is a set when X is.
- Does this help us show that $\Omega\Sigma X$ is a set when X is? (Not immediately)
- Is there a uniform definition of genuine finite multisets?
- What is $\mathrm{UP}(\mathbb{S}^1)$?
- If X is an n-type, is $\mathrm{UP}(X)$?
- Are there interesting operations $\mathrm{UP}(X) \to X$ (ultra-commutative)?

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Thank you!

$$\begin{tikzcd} \mathrm{Psh}(\mathrm{Orb}_{\Sigma_n}) \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathrm{Psh}(\mathrm{Orb}_{\Sigma_{n+1}}) \arrow[l] \end{tikzcd}$$