We now begin to explore synthetic homotopy theory: The study of the higher-dimensional properties of homotopy types using type theory. The tools are mostly the same as from classical homotopy theory: homotopy groups, co-/homology theories, etc.

We need at least one new type constructor: the pushout. This is an instance of a higher inductive type (HIT), and we begin by looking at a special case, namely the *circle* S^1. This nicely illustrates the homotopy hypothesis (recall: homotopy types = ∞-groupoids): S^1 is the homotopy type of the topological circle (the set $\mathbb{S}^1 :\equiv \{(x,y) : \mathbb{R}^2 \mid x^2+y^2=1\}$), and it is also the free ∞-groupoid on one object $\mathrm{base} : \mathrm{S}^1$ and one identification $\mathrm{loop} : \mathrm{base} =_{\mathrm{S}^1} \mathrm{base}$.

NB On Tuesday, the lecture/exercise class is cancelled so you can go to the talk by Johan M. Commelin about *formalizing perfectoid spaces* in Lean: it's 15:00–16:15 in S215-401.

The circle as a HIT

As a HIT, the circle is defined by the two constructors:

$$\mathrm{base} : \mathrm{S}^1$$
$$\mathrm{loop} : \mathrm{base} =_{\mathrm{S}^1} \mathrm{base}$$

Here, the constructor loop is called a *path constructor* because the codomain type is not S^1 itself, but rather a path/identification type of S^1. We mostly follow the discussion of § 15 in *Introduction to Homotopy Type Theory.*

A couple of remarks: We can phrase the induction principle as follows: Given a family P over S^1, we need data for the two constructors. For base, we need an element $p_{\mathrm{base}} : P(\mathrm{base})$. For the path constructor loop, we need not an element of P over loop (that wouldn't make sense), but instead a dependent path, aka a path over loop, $p_{\mathrm{loop}} : p_{\mathrm{base}} =^P_{\mathrm{loop}} p_{\mathrm{base}}$. Then we get a section

$$f :\equiv \mathrm{ind}^P_{\mathrm{S}^1}(p_{\mathrm{base}}, p_{\mathrm{loop}}) : \prod_{(x:\mathrm{S}^1)} P(x).$$

In the notes, the computation rules only hold up to a path. Note however, that to simplify matters, we assume $f\,\mathrm{base} \equiv p_{\mathrm{base}} : P(\mathrm{base})$. This is also the convention in the HoTT book. In cubical type theories, we can also assume a judgmental equality for the path constructor. Here, we only have an element

$$\mathrm{apd\text{-}ind}^P_{\mathrm{S}^1}(p_{\mathrm{base}}, p_{\mathrm{loop}}) : \mathrm{apd}_f\,\mathrm{loop} = p_{\mathrm{loop}},$$

both endpoints being elements of the paths-over type $p_{\mathrm{base}} =^P_{\mathrm{loop}} p_{\mathrm{base}}$.

The universal cover of the circle

Using the induction principle for S^1, we define the universal cover of the circle, $E : \mathrm{S}^1 \to \mathcal{U}$, such that $E(\mathrm{base}) :\equiv \mathbb{Z}$ and $\mathrm{tr}_E\,\mathrm{loop} \sim \mathrm{succ}_{\mathbb{Z}}$. We prove that the total type $\sum_{(x:\mathrm{S}^1)} E(x)$ is contractible. (We picture E as a helix lying over the circle.) As a corollary, we get that S^1 is a 1-type and $\Omega(\mathrm{S}^1, \mathrm{base}) \simeq \mathbb{Z}$.

The circle as the type of $\mathbb{Z}$-torsors

Recall from Lecture 8 that every group G arises as the automorphism group of an element $* : BG$, where BG is the classifying type of G. This is a pointed, connected 1-type obtained as the groupoid corresponding to the strict groupoid on one object $*$ with $\hom(*,*) :\equiv G$.

Just as we described BC_n for the cyclic group C_n as the type of cyclically ordered n-element sets, we can describe $B\mathbb{Z}$ as the type

$$\sum_{(X:\mathcal{U})}\sum_{(f:X\to X)}\|(X,f)=(\mathbb{Z},\mathrm{succ})\|,$$

of *infinite cyclically ordered sets*. These are also known as $\mathbb{Z}$-*torsors*.

We have a map $\varphi : \mathrm{S}^1 \to B\mathbb{Z}$ defined by induction by setting $\varphi(\mathrm{base}) :\equiv *$ and with $\mathrm{ap}_\varphi\,\mathrm{loop} : * =_{B\mathbb{Z}} *$ corresponding to the generator $1 : \mathbb{Z}$ under the equivalence $(* =_{B\mathbb{Z}} *) \simeq \mathbb{Z}$. As a surjective embedding, φ is an equivalence. (This uses that S^1 is connected, cf. exercises.)

It follows that $B\mathbb{Z}$ satisfies the universal property of the circle. This can also be shown without using S^1, cf. [1].

Exercises

- Prove that S^1 is connected: $\|\mathrm{S}^1\|_0$ is contractible.
- Define the multiplication map $- \cdot - : \mathrm{S}^1 \to \mathrm{S}^1 \to \mathrm{S}^1$ by double induction, corresponding to the multiplication of unit length complex numbers $\mathbb{S}^1 \subseteq \mathbb{C}$, and derive its basic laws:
  - Unit laws: $x \cdot \mathrm{base} = x$ and $\mathrm{base} \cdot x = x$ for $x : \mathrm{S}^1$.
  - Commutativity: $x \cdot y = y \cdot x$ for $x, y : \mathrm{S}^1$.
  - Inverse laws: $\bar{x} \cdot x = \mathrm{base}$ and $x \cdot \bar{x} = \mathrm{base}$ for $x : \mathrm{S}^1$, where $\bar{-} : \mathrm{S}^1 \to \mathrm{S}^1$ corresponds to complex conjugation.
  - What higher laws do these laws satisfy?
- In a previous exercise, we defined a group structure on any type of the form $\pi_1(X, x) :\equiv \|\Omega(X, x)\|_0$. Show that the group $\pi_1(\mathrm{S}^1, \mathrm{base})$ is isomorphic to $\mathbb{Z}$ with addition.

References

[1] Marc Bezem, Ulrik Buchholtz, and Daniel R. Grayson. *Construction of the Circle in UniMath*. Preprint. 2019. arXiv: 1910.01856.

Remember, there's no exercise session this week:
Instead, I wish you a Merry Christmas and a Happy New Year!
So think about the exercises over the break?