Introduction to Homotopy Type Theory WiSe19/20

Part I, Lecture 12 Ulrik Buchholtz

This week we continue with pushouts and related constructions. But we start with an application of pullbacks, namely we characterize the universes a object classifiers. Then we establish the main properties of pullbacks and other colimits.

Universes as object classifiers

If we assume propositional resizing, then the type of propositions, $\Omega :\equiv \mathrm{Prop}$, is a subobject classifier: For every embedding $f : X \to Y$, the type of pullback squares

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{1} \\ {\scriptstyle f}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \mathrm{true}} \\ Y & \longrightarrow & \Omega \end{array}$$

is contractible. Note that $\mathbb{1} \simeq \sum_{(P:\Omega)} P$.

Every (univalent) universe $\mathcal{U}$ is an object classifier: For every map $f : X \to Y$, the type of pullback squares

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{U}_* \\ {\scriptstyle f}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \mathrm{pr}_1} \\ Y & \longrightarrow & \mathcal{U} \end{array}$$

where $\mathcal{U}_* :\equiv \sum_{(X:\mathcal{U})} X$, is a proposition. (This proposition expresses that the fibers of f are $\mathcal{U}$-small.) The converse statement, that if a universe is an object classifier, then it is univalent, also holds.

We can use the notion of object classier to define the notion of *elementary* $(\infty, 1)$-*topos* externally, cf. [2], as a locally cartesian closed $(\infty, 1)$-category with finite limits and colimits, a subobject classifier, and enough object classifiers that classify classes of morphisms closed under composition, dependent products, and fiberwise limits and colimits.

Pushouts: More examples, flattening, and descent

In an $(\infty, 1)$-topos, colimits are *van Kampen*, meaning they are *universal* (stable under pullback) and satisfy *descent*. We can prove these properties internally for pullbacks, and for other colimits that can be defined in terms of pushouts (sequential colimits, coequalizers, etc.).

(Fun fact: A locally presentable $(\infty, 1)$-category is an $(\infty, 1)$-topos just when *all* colimits are van Kampen, cf. [1])

A very useful internal version of universality is the so-called *flattening lemma* that says that the total space of a type family over a colimit is again a colimit of the same shape over a diagram formed of total spaces. We'll see an application of this in the Hopf construction next week.

From pushouts we can construct many other finite colimits, such as (reflexive) coequalizers, sequential colimits, etc., and these again satisfy universality and descent.

Exercises

- Ex. 20.1 (the torus as a pushout).
- Ex. 20.3 (the fiberwise join).
- Ex. 20.5 (fiber of the wedge inclusion).
- Ex. 20.7 (pushout along an embedding).

References

[1] Urs Schreiber and Mike Shulman. *van Kampen colimit.* Aug. 14, 2013. URL: https://ncatlab.org/nlab/show/van+Kampen+colimit.

[2] Mike Shulman. *Elementary* $(\infty, 1)$*-Topoi.* Apr. 4, 2017. URL: https://golem.ph.utexas.edu/category/2017/04/elementary_1topoi.html.