Now that we've covered the basics of finite limits and colimits (in particular, pullbacks and pushouts), we're ready to do some homotopy theory. In particular, we'll define the higher homotopy groups $\pi_n(X, x_0)$ of a pointed type, and we'll work towards calculating $\pi_3(\mathbb{S}^2)$. This combines the long exact sequence of a pointed map with the Hopf fibration. We'll also need n-truncations and the notion of n-connected for arbitrary n. (We've already seen these for $n = -1, 0$.)

For reference, you can read §§ 28 and 31 in *Introduction to Homotopy Type Theory*, and/or Ch. 8 of the HoTT Book.

Pointed types and the suspension-loop space adjunction

Many of the algebraic invariants we shall consider, and in particular the higher homotopy groups, pertain to pointed types. Recall that $\mathcal{U}_* :\equiv \sum_{(X:\mathcal{U})} X$. The ($\mathcal{U}$-small) pointed types form an $(\infty, 1)$-category, of which just consider the low-dimensional structure: We define the type of pointed maps, $(X \to_* Y) :\equiv \sum_{(f:X\to Y)}(f * =_Y *)$, etc.

Of particular important to us is that the loop space and the suspension constructions form an adjunction $\Sigma \dashv \Omega$ on pointed types. As a corollary, $(\mathbb{S}^n \to_* X) \simeq_* \Omega^n X$.

n-Truncation and n-connectedness

We can define the $(n+1)$-truncation of a type $X : \mathcal{U}$ as the ($\mathcal{U}$-small) image of the map

$$\lambda x.\, \lambda y.\, \|x = y\|_n : X \to (X \to \mathcal{U}^{\leq n}),$$

note that $\mathcal{U}^{\leq n}$ is locally $\mathcal{U}$-small.

We say that a type X is n-connected if $\|X\|_n$ is contractible. A map is n-connected if all its fibers are. A key point is that $\pi_k(X, x)$ is trivial if X is n-truncated and $k > n$, or if X is n-connected and $k \leq n$.

Exercises

- (HoTT Book ex. 7.6) Prove that for $n \geq -1$, a type A is n-connected if and only if it is merely inhabited and for all $a, b : A$ the type $a =_A b$ is $(n-1)$-connected. Thus, since every type is (-2)-connected, n-connectedness of types can be defined inductively using only propositional truncations. (In particular, A is 0-connected if and only if $\|A\|$ and $\prod_{(a,b:A)} \|a = b\|$.)
- (HoTT Book ex. 8.6) For any pointed type A, let $i_A : \Omega A \to \Omega A$ denote inversion of loops, $i_A :\equiv \lambda p.\, p^{-1}$. Show that $i_{\Omega A} : \Omega^2 A \to \Omega^2 A$ is equal to $\Omega(i_A)$.
- (IntroHoTT ex. 29.2) Let $f : A \to_* B$ be a pointed map between pointed n-connected types, for $n \geq 1$. Show that the following are equivalent:
  1. f is an equivalence.
  2. $\Omega^{n+1}(f)$ is an equivalence.