Introduction to Homotopy Type Theory — WiSe19/20

Part I, Lecture 14 — Ulrik Buchholtz

We're working towards proving that $\pi_3(\mathbb{S}^2) \simeq \mathbb{Z}$. Our main tools are the long exact sequence (LES) of homotopy groups (HoTT Book § 8.4), and the Hopf construction (HoTT Intro § 31, giving us the homotopy type of the Hopf fibration $\mathbb{S}^3 \to \mathbb{S}^2$).

We need two other small ingredients, which we treat without proof: The Blakers-Massey theorem, and the associativity of the join operation.

Actually, we only need a simple corollary of the Blakers-Massey theorem, namely the Freudenthal suspension theorem. This can be proven directly, as in the HoTT Book, but it's good to know the statement of the former anyway.

The long exact sequence

A fiber sequence $F \xrightarrow{i} E \xrightarrow{p} B$ consists of pointed maps together with a pullback square:

$$\begin{array}{ccc} F & \xrightarrow{i} & E \\ \downarrow & & \downarrow p \\ \mathbb{1} & \longrightarrow & B. \end{array}$$

For any such fiber sequence we get a long exact sequence of homotopy groups:

$$\begin{array}{l} \cdots \\ \pi_n(F) \xrightarrow{\pi_n(i)} \pi_n(E) \xrightarrow{\pi_n(p)} \pi_n(B) \\ \cdots \\ \pi_1(F) \xrightarrow{\pi_1(i)} \pi_1(E) \xrightarrow{\pi_1(p)} \pi_1(B) \\ \pi_0(F) \xrightarrow{\pi_0(i)} \pi_0(E) \xrightarrow{\pi_0(p)} \pi_0(B) \end{array}$$

The Hopf construction

For us, *an H-space* is a pointed type A (writing $e :\equiv *_A$) together with a binary operation $\mu : A \to A \to A$ along with homotopies

$$\begin{aligned} \mathrm{leftunit}_A &: \mu(e, -) \sim \mathrm{id} \\ \mathrm{rightunit}_A &: \mu(-, e) \sim \mathrm{id} \\ \mathrm{cohunit}_A &: \mathrm{leftunit}_A\, e = \mathrm{rightunit}_A \end{aligned}$$

We say that the H-space A is *group-like* if the maps $\mu(x, -)$ and $\mu(-, x)$ are equivalences for all $x : A$. Any connected H-space is automatically group-like.

The Hopf construction associates to any group-like H-space A a fiber sequence $A \to A * A \to \Sigma A$.

Using the H-space structure on $\mathbb{S}^1$, along with associativity of the join, we get a fiber sequence $\mathbb{S}^1 \to \mathbb{S}^3 \to \mathbb{S}^2$. This is the Hopf fibration (in its homotopical incarnation).

Exercises

- Check that the type of H-spaces is equivalent to $\sum_{(A:\mathcal{U}_*)}(A \to_* (A \to_* A))$.
- Show that there is a fiber sequence $\mathbb{S}^3 \to \mathbb{S}^2 \to \|\mathbb{S}^2\|_2$.
- Use the natural H-space structure on $\mathbb{S}^0$ to construct a fiber sequence $\mathbb{S}^0 \to \mathbb{S}^1 \to \mathbb{S}^1$.