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This week we're reading section I.2 about dependent function types. Time permitting, we'll start section I.3 about the natural numbers.

Local and global forms of rules

For the type theory we're defining, all the rules can be used in any context. Because of this, we can also write the rules in *local style*. For example, we can write the Π-formation rule as

$$\frac{x : A \vdash B(x) \text{ type}}{\vdash \prod_{(x:A)} B(x) \text{ type}} \quad \text{and } \textit{mean} \quad \frac{\Gamma, x : A \vdash B(x) \text{ type}}{\Gamma \vdash \prod_{(x:A)} B(x) \text{ type}} \quad \text{for any } \Gamma.$$

This saves chalk, and perhaps makes better sense of the substitution convention: we write one fresh variable, x, and we write $B(x)$ to indicate that the schematic variable B depends on one more variable than whatever variables are already in the context. In the conclusion, the declaration $x : A$ is gone, indicating that x is *bound* in the expression $\prod_{(x:A)} B(x)$.

Categorical and hypothetical forms of rules

Some rules can be given in both *categorical* and *hypothetical* forms. For instance, for Π-elimination (aka evaluation or application), these are

$$\frac{\vdash f : \prod_{(x:A)} B(x) \qquad \vdash a : A}{\vdash f(a) : B(a)} \quad \text{resp.} \quad \frac{\vdash f : \prod_{(x:A)} B(x)}{x : A \vdash f(x) : B(x)} .$$

Given the structural rules (esp. substitution), these are equivalent. The benefit of the latter is that it's convenient for giving models (model theory), while the benefit of the former is that when all rules are in this style, then substitution becomes admissible (proof theory). (A rule is *admissible* if, whenever the premises are derivable, so is the conclusion.)

(Warning: Here, *categorical* is used as an antonym for hypothetical, but it's the *hypothetical* form that is closer to *category theory*.)

Judgmental extensionality from η

If A type, $x : A \vdash B(x)$ type, and $f, g : \prod_{(x:A)} B(x)$, then we have the following derived rule, which we think of as a judgmental extensionality principle:

$$\frac{x : A \vdash f(x) \equiv g(x) : B(x)}{\vdash f \equiv g : \prod_{(x:A)} B(x)}$$

Exercises

1. Check that the categorical and hypothetical forms of Π-elimination above are equivalent.
2. Give a derivation of the judgmental extensionality principle.
3. Exercises 2.3–2.5 from Rijke's notes (constant functions, dependent composition, argument-swap function).