We've finished defining our type theory! Now we can begin to develop the foundations of mathematics from the HoTT point of view. This means beginning with the concepts of equivalences and contractible types. We'll also study homotopies between and fibers of maps between types. (We're reading Sections 7 and 8 in *Introduction to Homotopy Type Theory.*) Contractible types are the simplest types of all, even simpler than propositions and sets, which we'll get to next week.

Proof assistant tutorials

- If you haven't already, check out the Natural Number Game: `http://wwwf.imperial.ac.uk/~buzzard/xena/natural_number_game/` (or `http://tinyurl.com/natgame1234`) – this introduces the Lean proof assistant in a fun way.
- Today, if the equipment works, I'll introduce Agda. See also Martín Escardó's *Introduction to Univalent Foundations of Mathematics with Agda* : `https://www.cs.bham.ac.uk/~mhe/HoTT-UF-in-Agda-Lecture-Notes/HoTT-UF-Agda.html`

Some references about universes

- The HoTT book [4] describes the usual external hierarchy of universes $\mathcal{U}_0 : \mathcal{U}_1 : \cdots$
- The superuniverse (a universe closed under a universe operator) was proposed by Palmgren [1] and its strength studied by Rathjen [2, 3].

Paths over paths

Given a type A, a type family $x : A \vdash B(x)$ type, two points $a, a' : A$, a path $p : a = a'$, and two elements of the fibers $b : B(a)$ and $b' : B(a')$, we define the type of *paths over the path* p:

$$(b =^B_p b') :\equiv (\mathrm{tr}_B\, p\, b =_{B(a')} b') \text{ type}$$

Observational equality types

We've already seen several examples of type family $\mathrm{Eq}_A : A \to A \to \mathcal{U}$, for some small type $A : \mathcal{U}$, namely for $A \equiv 2$, and $\mathbb{N}$. In this lecture, we'll define such a family for Σ-types and prove that $\mathrm{Eq}_\Sigma\, z\, w \simeq (z = w)$ for $z, w : \sum_{(x:A)} B(x)$.

In fact, for all inductive types, we can prove give such an reflexive observational equality family and prove that it is equivalent to the corresponding identity type.

Of the types in our type theory, that leaves Π-types, the universes $\mathcal{U}$, and of course the identity type itself. The observational equalities for Π-types and universes are homotopies and equivalences, respectively. For identity types, it depends on the ambient type.

Exercises

As always, the numbers refer to Rijke's *Introduction to Homotopy Type Theory.*

1. Ex. 7.1 some equivalences.
2. Ex. 7.3 (we set $A \leftrightarrow B :\equiv (A \to B) \times (B \to A)$).
3. Ex. 7.4 (the 3-for-2 property for equivalences).
4. Ex. 7.8 (on retractions).

References

[1] Erik Palmgren. "On universes in type theory". In: *Twenty-five years of constructive type theory (Venice, 1995)*. Vol. 36. Oxford Logic Guides. Oxford Univ. Press, New York, 1998, pp. 191–204.

[2] Michael Rathjen. "The strength of Martin-Löf type theory with a superuniverse. I". In: *Arch. Math. Logic* 39.1 (2000), pp. 1–39. DOI: 10.1007/s001530050001.

[3] Michael Rathjen. "The strength of Martin-Löf type theory with a superuniverse. II". In: *Arch. Math. Logic* 40.3 (2001), pp. 207–233. ISSN: 0933-5846. DOI: 10.1007/s001530000051.

[4] Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study: http://homotopytypetheory.org/book/, 2013.