This week we'll discuss the formalization of structural mathematics, including the beginnings of the algebraic hierarchy and a bit about categories (§ 14).

The Structure Identity Principle

We aim in type theory to capture each *type of mathematical object* as a *type in the sense of type theory* A, such that the mathematically natural identifications between $x, y : A$ correspond to elements of the identity type $x =_A y$. We call this the *structure identity principle*. Most types of *structures* are then 1-types/groupoids where the identifications are some kind of *isomorphism*. There are also 2-types, and more recently, non-truncated types such as that of $(\infty, 1)$-categories (which we don't know how define in type theory yet!).

trunc. level	type	identifications
0	Prop	$P \leftrightarrow Q$
0	$\mathbb{R}$	$x \leq y \wedge y \leq x$
1	Set	bijections
1	Monoid	bijections
1	MetricSpace	isometries
1	⋮	⋮
2	Groupoid	equivalences
2	Category	(adjoint) equivalences
2	†-Category	equivalences
⋮	⋮	⋮
∞	∞-Groupoid	equivalences
∞	$(\infty, 1)$-Category	equivalences

See [1] for further discussion.

Stuff, structure, and properties

Truncation levels help us make sense of another phenomenon of mathematical practice: Consider a map $f : A \to B$. We say that:

- f *forgets at most structure* if all the fibers are sets;
- f *forgets properties* if all the fibers are propositions;
- f *forgets nothing* if all the fibers are contractible.

(Of course, in the latter case, f is an equivalence.)

If f is n-truncated, we say that f *forgets at most n-stuff*.

We discuss some examples from the algebraic hierarchy.

Exercises

As always, the numbers refer to Rijke's *Introduction to Homotopy Type Theory.*

- Ex. 13.3: Functions are equivalent to functional relations.
- Ex. 13.5: Pointed equivalences.
- Show that composition of functions have witnesses of associativity satisfying the pentagon relation.
- For a fixed universe $\mathcal{U}$ and a type $A : \mathcal{U}$, show that taking fibers gives an equivalence

$$\sum_{(B:\mathcal{U})}\sum_{(f:B\to A)}\text{is-emb}(f) \simeq (A \to \mathrm{Prop}_{\mathcal{U}}).$$

- Have you seen other examples of 2-types in your mathematical studies so far?

References

[1] Steve Awodey. "Structuralism, Invariance, and Univalence". In: *Categories for the Working Philosopher*. Ed. by Elaine Landry. Oxford University Press, 2017. DOI: 10.1093/oso/9780198748991.003.0004.