We now take a little break from *Introduction to Homotopy Type Theory* to complete our discussion of the formalization of mathematics in HoTT/UF. We need therefore at least the propositional truncation type constructor (§ 23), which also allows us to construct set quotients (§ 24).

Propositional truncation

We've talked about the importance of propositions: because any two elements $p, q : P$ of a proposition P can be uniquely identified, it doesn't matter so much how we prove propositions, that is to say, construct elements of them.

However, in the pure constructive logic, not all the connectives preserve propositions. The exceptions are (1) disjoint union (pure disjunction), because $P + Q$ is not a proposition when both P and Q are true propositions, and (2) dependent sum types (pure existence), because $\sum_{(x:A)} P(x)$, for a propositional family P, is not a proposition if there are distinct $x, y : A$ with $P(x)$ and $P(y)$ both true.

A consequence of this is also that, as we have mentioned, we cannot say that a function $f : A \to B$ is *surjective*: the type $\prod_{(y:B)} \sum_{(x:A)} (f\,x = y)$ expresses rather that f is *split surjective*: it is equivalent (by type-theoretic choice) to $\mathrm{sec}(f)$, the type of sections of f.

Another consequence is that we cannot state the intent of the usual axiom of choice, one equivalent formulation of which is that any surjective function (merely) has a section.

Therefore, we introduce a new type-former, called *propositional truncation*. This gives for any type A another type $\|A\|$, that is a proposition, and a map $\eta_A : A \to \|A\|$ satisfying the universal property that for any proposition P, the precomposition map

$$- \circ \eta_A : (\|A\| \to P) \to (A \to P) \tag{$*$}$$

is an equivalence. Note that since both of the types in $(*)$ are propositions, this just says that there's a map in the other direction, i.e., to prove a proposition P from the hypothesis $\|A\|$, we may assume we have an element $a : A$.

The proposition $\|A\|$ expresses that A is *merely inhabited*: There are elements of A, but to get our hands on any one of them, we have (a priori) to be proving a (mere) proposition.

The logic of (mere) propositions

With the propositional truncation at our disposal, we can now consider the logic of propositions. We use the adjective *mere* to distinguish it from the *pure* constructive logic from Lecture 3:

logical constant	pure logic	mere logic
implication	$A \to B$	$A \to B$
conjunction	$A \times B$	$A \wedge B :\equiv A \times B$
disjunction	$A + B$	$A \vee B :\equiv \|A + B\|$
universal quant.	$\prod_{(x:A)} B(x)$	$\forall_{(x:A)} B(x) :\equiv \prod_{(x:A)} B(x)$
existential quant.	$\sum_{(x:A)} B(x)$	$\exists_{(x:A)} B(x) :\equiv \|\sum_{(x:A)} B(x)\|$
equality	$a =_A b$	$\|a =_A b\|$

Here we assume, except in the bottom row, that the ingredients in the right column are already propositions, so we only have to truncate for disjunctions and existential quantifications.

When we want to emphasize that we are using mere logic / propositional truncation, we can also use the adverb *merely*, so "there merely exists $a : A$ with $B(x)$" refers to the proposition $\exists_{(x:A)} B(x)$.

Assumption: From now on we assume that all universes $\mathcal{U}$ are closed under propositional truncation.

Theorem 1 (Induction principle for propositional truncation)**.** *Suppose P is a subtype of $\|A\|$, i.e., a propositional family over $\|A\|$, and suppose that $P(\eta_A\, x)$ holds for each $x : A$. Then $P(z)$ holds for each $z : \|A\|$.*

Proof. We have a section $p : \prod_{(x:A)} P(\eta_A\, x)$, so we get a map $\lambda x.\, (\eta_A\, x, p\, x) : A \to \sum_{(z:\|A\|)} P(z)$. The codomain is a proposition as a Σ-type of propositions over a propositions, so we get an induced map $s : \|A\| \to \sum_{(z:\|A\|)} P(z)$. Since $\|A\|$ is a proposition, this map is a section of $\mathrm{pr}_1 : \sum_{(z:\|A\|)} P(z) \to \|A\|$, so we get a section in $\prod_{(z:\|A\|)} P(z)$, as desired. □

In the HoTT book, the propositional truncation is introduced as a *higher inductive type*, with constructors $\eta_A : A \to \|A\|$ and $\mathrm{tr} : \prod_{(z,w:\|A\|)}(z = w)$, where the latter is a (recursive) *path constructor*. We then get that the section in the conclusion of Theorem 1 satisfies a computation rule, but we shan't need this.

We remark that the universal property of the propositional truncation exhibits the subuniverse $\mathrm{Prop}_{\mathcal{U}}$ of propositions as a *reflective subuniverse* of $\mathcal{U}$ (indeed a modality). This means that $\|{-}\|$ behaves as a left adjoint (in the $(\infty,1)$-categorical sense) to the inclusion of $\mathrm{Prop}_{\mathcal{U}}$ in $\mathcal{U}$. In particular, it is functorial:

Theorem 2. *For any map $f : A \to B$, there is map $\|f\| : \|A\| \to \|B\|$ giving rise to a naturality square*

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & \|A\| \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \|f\|} \\ B & \xrightarrow[\eta_B]{} & \|B\|. \end{array}$$

This comes from the universal property of η_A. Any such square commutes in a unique way since $\|B\|$ is a proposition.

Theorem 3. *For any type A and any type family B over A, the family of maps $\eta_{B(x)}$ induces an equivalence*

$$\Big\|\textstyle\sum_{(x:A)} B(x)\Big\| \to \Big\|\textstyle\sum_{(x:A)} \|B(x)\|\Big\|.$$

Proof. It suffices to construct a map from right to left. By the universal property, we may assume we have $z : \sum_{(x:A)} \|B(x)\|$. Using Σ-induction, we may assume we have $x : A$ and $w : \|B(x)\|$. By a second use of the universal property, we may assume we have $y : B(x)$, and then we have $\eta\,(x,y) : \Big\|\sum_{(x:A)} B(x)\Big\|$, as desired. □

The image factorization

The propositional truncation allows us to say that a map $f : A \to B$ is surjective; $\text{is-surj}(f) :\equiv \prod_{(y:B)} \|f^{-1}\, y\|$. It also allows us to factor any map as a surjection followed by an embedding. We need:

Lemma 4. *Let A be a type with type families B and C over A. If $f : \prod_{(x:A)}(B(x) \to C(x))$ is a family of surjections, then the induced map on total spaces* $\mathrm{tot}(f) : \sum_{(x:A)} B(x) \to \sum_{(x:A)} C(x)$ *is a surjection.*

This is immediate from the identification of the fibers of $\tilde{f}$ with fibers of each f_x.

Theorem 5. *For any $f : A \to B$ there is a unique factorization,*

$$\begin{array}{ccccc} A & & \xrightarrow{\quad f \quad} & & B \\ & {\scriptstyle g}\searrow & & \nearrow{\scriptstyle h} & \\ & & \mathrm{im}(f), & & \end{array}$$

where g is surjective and h is an embedding. More precisely, picking a universe $\mathcal{U}$ with $A, B : \mathcal{U}$, the type of such factorizations,

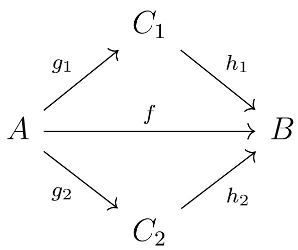
$$\textstyle\sum_{(C:\mathcal{U})}\sum_{(g:A\to C)}\sum_{(h:C\to B)}(h \circ g \sim f) \times \text{is-surj}(g) \times \text{is-emb}(h), \tag{†}$$

is contractible.

Proof. In a sense, this follows directly from the existence and universal property of the propositional trunction, applied in a context $y : B$, because f corresponds to the type $f^{-1}\, y$ in this context. Let us spell out the details.

We first describe the center of contraction: We take $\text{im}(f) :\equiv \sum_{(y:B)} \|f^{-1}\, y\|$, and let $g :\equiv \lambda x.\, (f\, x, \eta\, (x, \text{refl}_{f\, x}))$ and $h :\equiv \text{pr}_1$. Then h is an embedding, corresponding to the subtype $y \mapsto \|f^{-1}\, y\|$ of B, and g is equivalent to the map on total types corresponding to the family $\eta_{f^{-1}\, y}$, and is thus surjective by Lemma 4.

It remains to see that (†) is a proposition. Consider two factorizations



where g_1, g_2 are surjective and h_1, h_2 are embeddings. The fibers of h_1 and h_2 (at any $y : B$) are equivalent propositions, both equivalent to y being in the image of f. Thus, we get an induced equivalence $e : C_1 \simeq C_2$ that identifies g_1 with g_2, h_1 with h_2, and the homotopy $H_1 : h_1 \circ g_1 \sim f$ with $H_2 : h_2 \circ g_2 \sim f$. □

Locally small types and the replacement principle

All the models that we are interested in satisfy a very useful principle for the image factorization. To state it, we need the following notions, cf. Def. 24.5.1:

Definition 6. Fix a universe $\mathcal{U}$. A type A is *essentially $\mathcal{U}$-small* if there (purely) is a small type $X : \mathcal{U}$ and an equivalence $A \simeq X$. And A is *locally $\mathcal{U}$-small* if all the identity types are essentially $\mathcal{U}$-small.

In particular, for a locally $\mathcal{U}$-small type A, we have a reflexive family $\text{Eq}_A : A \to A \to \mathcal{U}$ inducing equivalences $(x =_A y) \simeq \text{Eq}_A\, x\, y$. Note that univalence implies that being essentially small, and hence also being locally small, are propositions.

For now we assume:

The Replacement Principle: For any map $f : A \to B$ from an essentially small type to a locally small type, the image $\text{im}(f)$ is essentially small.

Later we'll introduce the pushout types, and then we can prove the replacement principle.

Weakly constant maps

Sometimes, we want to construct an element of a set B from a hypothesis $\|A\|$, rather than just prove a proposition. For this, we need a map $f : A \to B$ that is *weakly constant*.

Definition 7. A map $f : A \to B$ is *weakly constant* if $\prod_{(x,x':A)}(f\, x =_B f\, x')$.

This is in contrast to a *constant* map, which is one that identifies with $\text{const}_y : A \to B$ for some $y : B$. Any constant map is indeed weakly constant. Note also that when B is a set, weak constancy of $f : A \to B$ is a proposition.

Theorem 8. *If $f : A \to B$ is a weakly constant map, and B is a set, then there is an induced map $g : \|A\| \to B$ such that $g \circ \eta_A = f$.*

Proof. Consider the image factorization of $f : A \xrightarrow{p} \mathrm{im}(f) \xrightarrow{i} B$. The key point is that $\mathrm{im}(f)$ is a proposition. Suppose $(y_1, z_1), (y_2, z_2) : \mathrm{im}(f)$. Since B is a set, the type $y_1 =_B y_2$ is a proposition. Hence we may hypothesize $x_1, x_2 : A$ with $f\, x_i = y_i$ for $i = 1, 2$. By concatenation, we get $y_1 = f\, x_1 = f\, x_2 = y_2$ and hence $(y_1, z_1) = (y_2, z_2)$.

By the universal property, we get $g' : \|A\| \to \mathrm{im}(f)$ such that $g' \circ \eta_A = p$. Composing with i we get $g :\equiv i \circ g' : \|A\| \to B$ with $g \circ \eta_A = i \circ g' \circ \eta_A = i \circ p = f$, as desired. □

Applying the theorem when B is A, we conclude that there is a "backwards" map $\|A\| \to A$ (A has *split support*) whenever A is a set with a weakly constant endomap. In fact, in this case, we don't need A to be a set:

Theorem 9. *Let A be a type. Then any weakly constant endomap $f : A \to A$ factors through $\eta_A : A \to \|A\|$.*

Proof. Since f is weakly constant, we have $H : \prod_{(x,y:A)}(f\, x = f\, y)$. This time we first factor f through its type of *fixed points*, $\mathrm{fix}(f) :\equiv \sum_{(x:A)}(f\, x = x)$. Indeed, we have $g :\equiv \lambda x.\, (f\, x, H\, (f\, x)\, x) : A \to \mathrm{fix}(f)$, and $h :\equiv \mathrm{pr}_1 : \mathrm{fix}(f) \to A$, and $h \circ g = f$.

So all we need to do is to prove that $\mathrm{fix}(f)$ is a proposition. We prove $\mathrm{fix}(f) \to \text{is-contr}(\mathrm{fix}(f))$, so let x_0 be a fixed point of f. Then concatenating with $H\, x_0\, x$ gives a family of equivalences $(f\, x = x) \simeq (f\, x_0 = x)$ for $x : A$, so we get

$$\mathrm{fix}(f) \equiv \textstyle\sum_{(x:A)}(f\, x = x) \simeq \sum_{(x:A)}(f\, x_0 = x) \simeq \mathbb{1}. \qquad \square$$

Here's another application of the idea, showing how to construct an element of a type B from a hypothesis $P \vee Q$.

Theorem 10. *Let P and Q be propositions and B an arbitrary type. Let $f : P \to B$, $g : Q \to B$ be functions with $H : \prod_{(p:P)} \prod_{(q:Q)} f\, p = g\, q$. Then $[f, g] : P + Q \to B$ factors through $P \vee Q$.*

Proof. This time we factor through the type

$$R :\equiv \textstyle\sum_{(y:B)} \sum_{(s:\prod_{(p:P)} f\, p = y)} \sum_{(t:\prod_{(q:Q)} g\, q = y)} \prod_{(p:P)} \prod_{(q:Q)} (H\, p\, q = s\, p \cdot (t\, q)^{-1}).$$

The map $R \to Y$ is simply the first projection. We construct a map $P + Q \to R$ by induction. We send $p : P$ to the tuple with $f\, p$ as the first component, the second component is $\mathrm{refl}_{f\, p}$ relative to the fact that P is contractible, the third component gives the inverse of H, and the fourth component is a law about inverses relative to the contractibility of P. The case for $q : Q$ is similar.

The key point is now that R is contractible assuming $P \vee Q$. We show this for some $p : P$ (again, the case $q : Q$ is similar). Then P is contractible, and we have

$$\begin{aligned} R &\simeq \textstyle\sum_{(y:B)} \sum_{(s:f\, p = y)} \sum_{(t:\prod_{(q:Q)} g\, q = y)} \prod_{(q:Q)} (H\, p\, q = s \cdot (t\, q)^{-1}) \\ &\simeq \textstyle\sum_{(t:\prod_{(q:Q)} g\, q = f\, p)} \prod_{(q:Q)} (t\, q = (H\, p\, q)^{-1}) \simeq \mathbb{1}. \end{aligned}$$

We then get a map $P \vee Q \to R$ such that the composite through to B factors $[f, g]$. □

Exercises

- Show that $\mathrm{LEM}_{\mathcal{U}}$ implies that for all $A : \mathcal{U}$ we have $\|\|A\| \to A\|$.
- (Ex. 23.14) Let $f : A \to B$. Show that f is an equivalence if and only if f is both surjective and an embedding.