Finite sets

As an application of Theorem 8 from yesterday's sheet, we consider the type of finite sets,

$$\mathrm{FinSet} :\equiv \textstyle\sum_{(X:\mathcal{U})} \exists_{(n:\mathbb{N})}(X \simeq \mathrm{Fin}\, n) \equiv \sum_{(X:\mathcal{U})} \|\sum_{(n:\mathbb{N})} X \simeq \mathrm{Fin}\, n\|.$$

Since having decidable equality is a proposition, and each $\mathrm{Fin}\, n$ has decidable equality, so does each finite set.

Theorem 1. *There is a function* $\mathrm{card} : \mathrm{FinSet} \to \mathbb{N}$ *that is a retraction of* $\mathrm{Fin} : \mathbb{N} \to \mathrm{FinSet}$.

Proof. Fix $X : \mathrm{FinSet}$. We know that there merely exists $n : \mathbb{N}$ and an equivalence $X \simeq \mathrm{Fin}\, n$, so it suffices to show that the first projection $\mathrm{pr}_1 : \sum_{(n:\mathbb{N})}(X \simeq \mathrm{Fin}\, n) \to \mathbb{N}$ is weakly constant. But this follows from trichotomy of the order on $\mathbb{N}$ together with the pigeonhole principle. □

We also conclude that FinSet is equivalent to the type $\sum_{(X:\mathcal{U})} \sum_{(n:\mathbb{N})} \|X \simeq \mathrm{Fin}\, n\|$, where the cardinality n is outside the truncation.

Note that the replacement principle implies that FinSet is essentially small, as the image of $\mathrm{Fin} : \mathbb{N} \to \mathcal{U}$. So the replacement principle is incompatible with models where types in the ith universe are i-truncated.

Quotients of equivalence relations

Suppose $A : \mathcal{U}$ is a type with an equivalence relation $R : A \to A \to \mathrm{Prop}_{\mathcal{U}}$ (a proposition-valued relation that is reflexive, symmetric, and transitive). We now show that propositional truncation, together with the replacement principle, allows us to construct the *quotient set* $A/R : \mathrm{Set}_{\mathcal{U}}$.

Theorem 2. *Given A and R as above, there is a small set $A/R : \mathrm{Set}_{\mathcal{U}}$ with a surjection $q_R : A \to A/R$ such that*

1. *$(q_R\, x = q_R\, x') \simeq (R\, x\, x')$ for all $x, x' : A$;*
2. *for any* set *B, a function $f : A \to B$ factors uniquely through q_R if $f\, x = f\, x'$ for all $x, x' : A$ with $R\, x\, x'$.*

Proof. Take A/R to be a small type equivalent to the image of the map $[-]_R : A \to \mathrm{Prop}_{\mathcal{U}}^A$, where $[x]_R\, y :\equiv R\, x\, y$. (The image is essentially small since A is assumed to be small, and $\mathrm{Prop}_{\mathcal{U}}$ is locally small.)

(1) is a special case of the fact that for any $P : \mathrm{Prop}_{\mathcal{U}}^A$, we have $([x]_R = P) \simeq P\, x$. This uses the fact that R is an equivalence relation.

(2) Uniqueness: If g, h are extensions of f through q_R, then for any $z : A/R$, the type $g\, z = h\, z$ is a proposition since B is a set, so we may assume $z \equiv q_R\, x$ for some $x : A$. Then $g\,(q_R\, x) = f\, x = h\,(q_R\, x)$, as desired.

Existence: Let $z : A/R$. To define the image of z in B it suffices to give a weakly constant map $\sum_{(x:A)}(z = q_R\, x) \to B$, and $f \circ \mathrm{pr}_1$ does the trick. □

For another proof, see Theorem 24.2.3 in the notes.

Set truncation

As an example of a set quotient, we can consider the *set truncation* of a type $A : \mathcal{U}$. This is a set $\|A\|_0$ with a map $\eta : A \to \|A\|_0$ satisfying the analogous properties of the propositional truncation $\|A\|_{-1} :\equiv \|A\|$. It is the set quotient of A with respect to the relation $\|x = y\|$ for $x, y : A$.

We say that a type A is *connected* if $\|A\|_0$ is contractible.

Poset and category completion

As a variation of the above, we can consider a pre-ordered set $(A, \leq)$, which we recall is a precategory with hom-propositions $x \leq y$. Here, isomorphism corresponds to the equivalence relation $R\,x\,y :\equiv (x \leq y \wedge y \leq x)$, and we can use the set quotient to give the *poset completion*, a poset $\overline{A}$ with a monotone map $A \to \overline{A}$ such that for every poset B, we precomposition map $\mathrm{Fun}(\overline{A}, B) \to \mathrm{Fun}(A, B)$ is an equivalence.

In this case, however, there is another image that does the trick, namely the image of the Yoneda embedding $A \to_{\mathrm{Fun}} \mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Prop}_{\mathcal{U}})$. We state this for general pre-categories, but first we need:

Definition 3. A functor $F : A \to B$ of precategories is *essentially surjective on objects* if for all $y : B$ there *merely* exists $x : A$ with $F\,x \cong y$. We say that F is a *weak equivalence* if it is fully faithful and essentially surjective on objects.

Theorem 4. *Let A be a small pre-category. Then there is a small category $\overline{A}$ together with a weak equivalence $\eta : A \to \overline{A}$.*

Proof. Consider the Yoneda functor from A to $\mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Set}_{\mathcal{U}})$. The underlying map on objects is a map from a small type to a locally small type, and we take the objects of $\overline{A}$ to be the small type equivalent to the image. As a category, $\overline{A}$ is then defined to be the full subcategory of $\mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Set}_{\mathcal{U}})$. □

(This operation is known as *Rezk completion* in the HoTT book.)

All groups are automorphism groups

As an application, fix a group G, and consider the precategory with $\mathbb{1}$ for objects, and $\hom(*, *) :\equiv G$, using the group operation as composition. Let BG be the (type of objects) of the corresponding category. (This category is a groupoid so no information is lost by just looking at the underlying 1-type/groupoid.)

It follows that there is an object $* : BG$, that BG is a connected 1-type, and we have a group isomorphism $(* =_{BG} *) \simeq G$. That is, we've exhibited G as the automorphism group $\mathrm{Aut}_{BG}(*)$. BG is known as the *classifying type* of G, and also as the Eilenberg-MacLane type $K(G, 1)$.

As a concrete example, consider the cyclic group C_n. Unraveling the above definition, we see that BC_n is equivalent to the type

$$\textstyle\sum_{(X:\mathcal{U})}\sum_{(f:X\to X)} \|(X, f) = (\mathrm{Fin}\,n, s)\|,$$

where $s : \mathrm{Fin}\,n \to \mathrm{Fin}\,n$ is the cyclic successor operation. So we can identify the elements of BC_n with *cyclically ordered n-element sets.*

Later, we'll show that the type of groups is equivalent to the type of pointed, connected 1-types.

n-truncations

We briefly give here the definition of the higher truncations. We'll verify the universal property later when we study them in depth. The n-truncation of $A : \mathcal{U}$ is the type $\|A\|_n : \mathcal{U}^{\leq n}$ defined by recursion on $n \geq -2$, where $\|A\|_{-2} :\equiv \mathbb{1}$, $\|A\|_{-1} :\equiv \|A\|$ (the propositional truncation), and for $k \geq -1$, $\|A\|_{k+1}$ is the small image of the map $\lambda x.\, \lambda y.\, \|x = y\|_k : A \to (A \to \mathcal{U}^{\leq k})$.

Exercises

- Prove that a functor $F : A \to B$ between categories is an equivalence if and only if it is a weak equivalence.
- (Ex. 24.2) Show that the set truncation of a loop space is a group.