Logical Aspects of Simplicial Type Theory

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Outline

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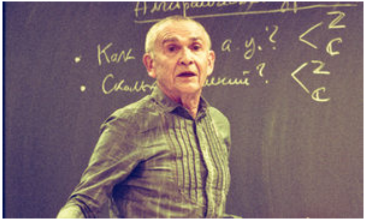
Homotopical mathematics

> *I am pretty strongly convinced that there is an ongoing reversal in the collective consciousness of mathematicians: the right hemispherical and homotopical picture of the world becomes the basic intuition, and if you want to get a discrete set, then you pass to the set of connected components of a space defined only up to homotopy.* Yuri Manin 2009 [Gel09]

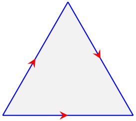
Work "up to homotopy" – unique existence becomes contractibility. For example:

- Spectral algebraic geometry – proofs of:
  - Milnor and Bloch–Kato conjectures by Rost & Voevodsky [Wei09]
  - Weibel conjecture [KST18]
  - Geometric Langlands by Gaitsgory et al. [Ari+24]
- Quantum field theory (higher gauge theory, Chern–Simons, . . .)



(Denis Mironov, CC BY-NC-ND)

The "dunce hat" is contractible but not collapsible:



Homotopy type theory

Homotopy type theory adds to Martin-Löf type theory

$$\Sigma, \Pi, =, 0, +, \mathbb{N}, \mathcal{U}_i, \ldots$$

Voevodsky's univalence axiom

$$\mathsf{isEquiv}(A =_{\mathcal{U}} B \to A \simeq B)$$

and homotopical generalizations of inductive types (pushouts, truncations, initial algebras, localizations, . . .)

Implementation of *Univalent Foundations* whose key feature is:

Univalence Principle

Equivalent mathematical structures are indistinguishable.

AKA equivalence/structure identity principle – Makkai, Aczel, Awodey, . . . [Ahr+25; Awo18]

(Renate Schmid, CC BY-SA)

Metatheory of HoTT, classically

Theorem (Voevodsky [KL21])

HoTT can be modeled using the Kan–Quillen model structure on simplicial sets with type families interpreted as Kan fibrations.

This model is the "standard model" generalizing the set theoretic model of MLTT – validates LEM, AC, Whitehead's principle,

$$(\Pi n : \mathbb{N}.\, \pi_n(X) = 0) \to X = 1,$$

sets cover, and any set theoretic principles of the metatheory.

Theorem (Shulman [Shu19])

Every Grothendieck $(\infty, 1)$-topos admits a presentation by type theoretic model topos, which interprets HoTT.

Classical metatheory: these models validates impredicative reasoning via *propositional resizing*,

$$\mathsf{isEquiv}(\mathsf{Prop}_0 \hookrightarrow \mathsf{Prop}_i)$$

where $\mathsf{Prop}_i := \Sigma X : \mathcal{U}_i.\, \Pi x, x' : X.\, x =_X x'$ is the type of propositions in $\mathcal{U}_i$.

Metatheory of HoTT, constructively

Theorem (Sattler [Sat25])

In predicative constructive set theory, we can build a model of HoTT using a uniform Kan model structure on cubical sets that is Quillen equivalent to the Kan model structure on semisimplicial sets.

Builds on line of work on cubical models of type theory by Coquand et al. [BCH14; Coh+18] that give normalizing cubical type theories. This model validates:

Pointwise Principle

A family of propositions has a section if it has one on points.

Get dependent choice, presentation axiom, . . . , but *not* propositional resizing.

Earlier versions already established that the consistency strength of predicative HoTT mirrors MLTT – from ATR_0 to KPM [Rat17] depending on the availability of inductive types.

Expected to lead to constructive models in higher toposes, realizability models, etc.!

Mathematics in HoTT

Quite a lot of mathematics has be developed in HoTT [Uni13], often formalized using computer proof assistants such as Agda, Rocq, Lean.

- Basically all "truncated" mathematics (at the level of sets, groupoids, 2-groupoids, . . . , perhaps with appropriate axioms)
- Basic results about arbitrary types, e.g., idempotent modalities [RSS20], Blakers–Massey [Hou+16; Ane+20], path spaces of pushout [Wär24]
- Theory of higher groups [BDR18; BR23]
- Cellular (co)homology, spectral sequences, . . .

But! We don't know how to define higher monoids, categories, . . . – it's an open problem:

Coherence problem

Can HoTT develop the theory of $(\infty, 1)$-categories capturing the higher category structure on each $\mathcal{U}_i$ with functions as morphisms?

Equivalently, can we internalize (*abstract*) the external sequence of truncated semisimplicial types

$$\mathsf{Fun}(\Delta^{\mathsf{op}}_{\leq n}, \mathcal{U}_i)$$

as a family $S : \mathbb{N} \to \mathcal{U}_{i+1}$?

Outline

Context: Homotopy type theory

Simplicial type theory

Multimodal type theory

Duality

Outlook

Simplicial type theory

Riehl–Shulman [RS17] introduced simplicial type theory. The idea is to interpret (homotopy) type theory in simplicial objects $s\mathcal{E}$ of a model of HoTT $\mathcal{E}$. Here we have the simplices Δ^n, generated from an interval type $\mathbb{I}$ (totally ordered with $0 \neq 1$).

Using $\mathbb{I}$, we get the type of arrows in any type, $X^{\mathbb{I}}$, and we get functoriality for free by composition: $\mathbb{I} \to X \to Y$.

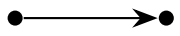
Definition X is *Segal* if $X^{\Delta^2} \to X^{\Lambda^2_1}$ is an equivalence.

Definition A Segal type X is *Rezk* if $X^{\mathbb{E}} \to X$ is an equivalence, for $\mathbb{E}$ the "walking equivalence".

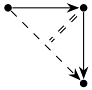
These model homotopical categories: In the model they corresponding to $(\infty, 1)$-categories in $\mathcal{E}$.

Definition Functions $f : C \to D$ and $g : D \to C$ are adjoint when equipped with $\iota : \Pi c, d.\, \mathsf{hom}(f(c), d) \simeq \mathsf{hom}(c, g(d))$.

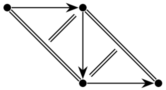
$\mathbb{I}$



$\Lambda^2_1 \hookrightarrow \Delta^2$



$\mathbb{E}$



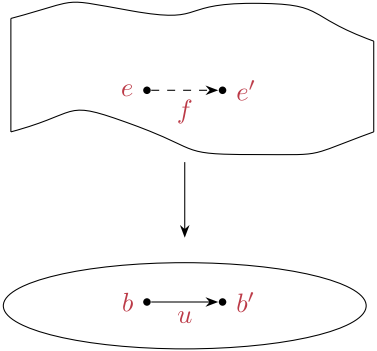
Further developments in simplicial type theory

The basic setup suffices for many purposes:

- Fibered category theory [BW23]
- (co)limits and exponentiable functors, Bardomiano Martínez [Bar22; Bar24a]
- Prototype proof assistant $\textsc{Rzk}$ with formalization of the fibrational Yoneda lemma [Kud23; KRW04]: For a covariant family $C : A \to \mathcal{U}$ over a Segal type A, evaluation at $a : A$ is an equivalence:

$$(\Pi x.\, \mathsf{hom}(a, x) \to C(x)) \to C(a)$$

A covariant family:



But it doesn't provide examples, most notably the category of spaces $\mathcal{S}$.

Directed univalence and the Yoneda embedding

In recent work with Daniel Gratzer and Jonathan Weinberger [GWB24; GWB25], we extend simplicial type theory to solve this: Construct the category of spaces $\mathcal{S}$ and the Yoneda embedding $y : C \to \widehat{C}$.

The required extensions are:

- Modalities: core groupoid $\flat$, opposite category op, and amazing right adjoint $(-)_{\mathbb{I}}$ (for constructing $\mathcal{S}$) and twisted arrow category tw (for constructing y).
- Cubes separate: A map $f :_{\flat} A \to B$ is an equivalence if it induces equivalences on $\langle \flat \mid \mathbb{I}^n \to A \rangle \to \langle \flat \mid \mathbb{I}^n \to B \rangle$
- Duality axiom (see below)

From $\mathcal{S}$, we get category of higher algebras, e.g., higher monoids Mon with *directed univalence principles*: the arrows in Mon are homomorphisms.

From $\flat$ we can state and derive *pointwise* principles, e.g., pointwise left adjoints are left adjoints.

Extended simplicial type theory

To construct $\mathcal{S}$ we need to enlarge our topos to cubical objects, $\mathsf{c}\mathcal{E}$ to ensure we have an adjunction $(-)^{\mathbb{I}} \dashv (-)_{\mathbb{I}}$. We also have a string of adjoint functors to the base model $\mathcal{E}$,

$$\begin{array}{c} \mathsf{c}\mathcal{E} \\ \Pi \Big\downarrow \dashv \Big\uparrow \dashv \Big\downarrow \dashv \Big\uparrow \nabla \\ \mathcal{E} \end{array}$$

On $\mathsf{c}\mathcal{E}$ we get $\flat \equiv \Delta\Gamma \dashv \nabla\Gamma \equiv \sharp$.

We want $\mathcal{S}$ to classify covariant fibrations, given by a family $\mathsf{isCov} : \mathcal{U}^{\mathbb{I}} \to \mathsf{Prop}$.

$$\mathcal{S} := \sum_{A:\mathcal{U}_{\mathsf{simp}}} \big(\mathsf{isCov}(\lambda i.\, A^{\eta}(i))\big)_{\mathbb{I}}$$

where $\mathcal{U}_{\mathsf{simp}}$ is the subuniverse of simplicial types, i.e., the types A with

$$\prod_{i,j:\mathbb{I}} \mathsf{isEquiv}(A \to A^{i\leq j \vee j \leq i})$$

Outline

Multimodal type theory

The need for modal type theory: We often have operations that don't preserve all connectives/constructions, or are not applicable in all contexts (fibered), e.g., $\flat, (-)_{\mathbb{I}}$, or:

$$\frac{\text{the number of planets is } 8 \qquad \text{it is necessary that } 8 = 8}{\text{it is necessary that the number of planets} = 8} \times$$

Multimodal type theory (MTT) [Gra+20] concerns a wide class of modal operators: the (dependent) right adjoints

Long history, much abridged:

- Dual-context calculi [PD01] – doesn't scale beyond one modality,
- Delayed substitutions [Bd00] – no hope of decidable type checking
- Fitch-style [GSB19]: assumes adjunction on modalities themselves

Dual-context introduction:

$$\frac{\Delta; \cdot \vdash M : A}{\Delta; \Gamma \vdash \mathsf{mod}_\mu(M) : \langle \mu \mid A \rangle}$$

With multiple modalities, annotate variables:

$$\Gamma, x :_\mu A$$

Now intro rule is hard!
Delayed substitution:

$$\frac{\Gamma \vdash \gamma : \mu\Gamma' \qquad \Gamma' \vdash M : A}{\Gamma \vdash \mathsf{mod}_\mu(M)^\gamma : \langle \mu \mid A \rangle^\gamma}$$

Multimodal type theory

MTT is parametrized by a *mode theory* $\mathcal{M}$, a 2-category with:

- objects m – modes
- morphisms $\mu : m \to n$ – modalities
- 2-cells $\alpha : \mu \Rightarrow \nu$ – natural maps of modalities

A model is (can be) a 2-functor $F : \mathcal{M} \to \mathsf{Cat}$ with each F_m locally cartesian closed and such that each F_μ has a left adjoint $L_\mu \dashv F_\mu$.

Theorem (Gratzer [Gra22])

If $\mathcal{M}$ is decidable, then MTT has normalization and type checking is decidable.

New intro rule for $\mu : m \to n$

$$\frac{\Gamma/\mu \vdash M : A \;@m}{\Gamma \vdash \mathsf{mod}_\mu(M) : \langle \mu \mid A \rangle \;@n}$$

where Γ/μ is interpreted using L_μ and $\langle \mu \mid A \rangle$ is interpreted using F_μ.
This makes sense if the formation rule is:

$$\frac{\Gamma/\mu \vdash A \;\mathsf{type}\;@m}{\Gamma \vdash \langle \mu \mid A \rangle \;\mathsf{type}\;@n \qquad \Gamma, x :_\mu A \;\mathsf{ctx}@n}$$

MTT: Variable and elimination rules

The variable rule comes from the counit:

$$\frac{\mu : m \to n}{\Gamma, x :_{\mu} A / \mu \vdash x : A \ @m}$$

This can relaxed to build in the action of 2-cells:

$$\frac{\mu, \nu : m \to n \qquad \alpha : \mu \Rightarrow \nu}{\Gamma, x :_{\mu} A / \nu \vdash x^{\alpha} : A^{\alpha} \ @m}$$

And build in weakening:

$$\frac{\mu : m \to n \qquad \alpha : \mu \Rightarrow \mathsf{mods}(\Gamma')}{\Gamma, x :_{\mu} A, \Gamma' \vdash x^{\alpha} : A^{\alpha} \ @m}$$

The elimination rule is pattern matching:

$$\frac{\begin{array}{c} \mu : m \to n \\ \nu : n \to o \qquad \Gamma, x :_{\nu} \langle \mu \mid A \rangle \vdash B \ \mathsf{type} \ @o \\ \Gamma / \nu \vdash M_0 : \langle \mu \mid A \rangle \ @n \\ \Gamma, y :_{\nu\mu} A \vdash M_1 : B[\mathsf{mod}_{\mu}(y)/x] \ @o \end{array}}{\mathsf{let}_{\nu}\ \mathsf{mod}_{\mu}(x) \leftarrow M_0 \ \mathsf{in}\ M_1 : B[M_0/x] \ @o}$$

From these we easily derive:

$$\mathsf{coe}_{\alpha} : \langle \mu \mid A \rangle \to \langle \nu \mid A \rangle$$

and

$$\mathsf{comp} : \langle \mu\nu \mid A \rangle \to \langle \mu \mid \langle \nu \mid A \rangle \rangle$$

Propositional reduct of MTT, other applications

See [KG23]: If the mode theory encodes a comonad $\square$ on a single mode, we get the modal logic intuitionistic $\mathrm{S4}$. We get the same for an *idempotent* comonad, but they differ at the level of proofs.

Other applications of MTT include guarded recursion with Löb induction [Gra25]:

$$\begin{tikzcd} t \arrow[loop left, "{\ell,e}"] \arrow[r, bend left, "\varepsilon"] \arrow[r, bend right, "\gamma"'] & s \arrow[l, "\delta"] \arrow[loop right, "\top"] \end{tikzcd}$$

Outline

The duality axiom

Blechschmidt introduced the following duality axiom [Ble23], with an eye to synthetic algebraic geometry:

Consider an algebraic (or Horn) theory $\mathbb{T}$. It has a classifying 1-topos $\mathsf{Set}[\mathbb{T}] = [\mathbb{T}\text{-}\mathsf{mod}_{\mathsf{fp}}, \mathsf{Set}]$ (whose enveloping $(\infty, 1)$-topos is $\mathcal{S}[\mathbb{T}] = [\mathbb{T}\text{-}\mathsf{mod}_{\mathsf{fp}}, \mathcal{S}]$, classifying set models of $\mathbb{T}$).

Internally in $\mathsf{Set}[\mathbb{T}]$, we have the universal $\mathbb{T}$-model $U_{\mathbb{T}}$ and a new theory of $U_{\mathbb{T}}$-algebras. It validates:

Duality

For any finitely presented $U_{\mathbb{T}}$-algebra A, the evaluation homomorphism $A \to (\mathsf{Spec}(A) \to U_{\mathbb{T}})$ is invertible.

Here, $\mathsf{Spec(A)} := \mathsf{hom}_{U_{\mathbb{T}}}(A, U_{\mathbb{T}})$

Proof idea. Follow the Kripke–Joyal translation. Since slicing over a stage T gives another classifying topos of the same form, $\mathsf{Set}[\mathbb{T}]/T \simeq \mathsf{Set}[\mathbb{T}/T]$, we may assume we're at the initial model, and have a finitely presented $\mathbb{T}$-model A. But for these it's easy to show duality by the Yoneda lemma.

Duality and synthetic mathematics

Some applications:

- Synthetic algebraic geometry [CCH24]: $\mathbb{T}$ is rings, $R := U_{\mathbb{T}}$ is modal for the topology of local rings, and then $x \neq 0 \leftrightarrow \mathsf{inv}(x)$ and $\neg(x \neq 0) \leftrightarrow \mathsf{nilp}(x)$.
- Synthetic Stone duality [Che+24] & light condensed mathematics [Bar24b]: $\mathbb{T}$ is boolean algebras, $U_{\mathbb{T}}$ is local for topology forcing every $r : U_{\mathbb{T}}$ either $r = 0$ or $r = 1$, duality for countably presented boolean algebras, get Markov's principle and LLPO.
- Synthetic domain theory [PS25]

Often get local choice principles [Wil25].

NB Close relation to Kock–Lawvere axiom for synthetic differential geometry. But the infinitary nature of the theory of C^{∞}-rings could hide a more general axiom.

In general, the quest is for a complete description of the non-geometric validities in the classifying topos $\mathsf{Set}[\mathbb{T}]$ of a geometric theory $\mathbb{T}$.

Duality in simplicial type theory

For simplicial type theory, we take $\mathbb{T}$ to be the theory of bounded distributive lattices and $\mathbb{I} := U_{\mathbb{T}}$.

Again, $\mathbb{I}$ is modal for the topology forcing $\prod_{i,j:\mathbb{I}}(i \leq j \vee j \leq i)$ and $0 \neq 1$ (by duality!), so duality descends to simplicial sets, classifying bounded total orders with distinct endpoints.

Lemma (Phoa principle)

Evaluation at $0, 1$ *is an embedding* $(\mathbb{I} \to \mathbb{I}) \to \mathbb{I} \times \mathbb{I}$ *with image* Δ^2*.*

Proof.

$\mathsf{Spec}(\mathbb{I}[x]) = \mathbb{I}$, so $\mathbb{I}[x] \simeq (\mathbb{I} \to \mathbb{I})$. Any polynomial in one variable has the normal form $p(x) = p(0) \vee x \wedge p(1)$. □

Generalized Phoa also holds:

- Evaluation gives an equivalence from $\mathbb{I}^n \to \mathbb{I}$ to $\mathsf{Pos}((0 < 1)^n, \mathbb{I})$.
- Evaluation gives an equivalence from $\Delta^n \to \mathbb{I}$ to $\mathsf{Pos}([0 \leq \cdots \leq n], \mathbb{I})$.

Corollary

Δ^n *is Rezk.*

Proof.

A composable pair of arrows gives n maps $[0 < 1 < 2] \to \mathbb{I}$, which by Phoa assemble to a composite $\Delta^2 \to \Delta^n$. □

Finally, we use Phoa's principle to prove directed univalence for $\mathcal{S}$.

Summary of results in modal simplicial type theory

- The universe $\mathcal{S}$ is a category (Rezk type) whose terms are groupoids (modally discrete).
- $\mathcal{S}$ classifies (amazingly) covariant families (left fibrations).
- $\mathcal{S}$ is closed under Σ, $=$, and finite colimits.
- $\mathcal{S}$ is directed univalent:

$$(\mathbb{I} \to \mathcal{S}) \simeq \sum_{A,B:\mathcal{S}} (A \to B)$$

- A natural transformation is invertible if it is objectwise invertible.
- There is a fully faithful functor $y : C \to \mathsf{PSh}(C) := (\langle \mathsf{op} \mid C \rangle \to \mathcal{S})$.
- $\mathsf{PSh}(C)$ is the free cocompletion of C.
- Formula for pointwise Kan extensions.
- Quillen's Theorem A: A functor $f :_\flat C \to D$ is right cofinal if and only if for all $d :_\flat D$, $L_{\mathbb{I}}(C \times_D D_{d/}) = 1$.
- Right cofinal maps are stable under pullback by cocartesian functors.
- If C has finite and filtered colimits, then it is cocomplete, using [SW25].
- The category of spectra $\mathcal{S}\mathrm{p} = \varprojlim(\mathcal{S}_* \leftarrow \mathcal{S}_* \leftarrow \dots)$ is stable and cocomplete, and the smash product is associative, using [Lju24]
- ...

Outline

Outlook

There's lots more to do with simplicial type theory:

- Construct a directed univalent category of categories.
- From this, almost all higher category theory and higher algebra should follow.
- Think about extracting a normalizing calculus from the constructive model.
- Feed this into the proof assistant RZK.
- Formalize in Agda, using modal extension (Sam Toth).
- Embed simplicial type theory in other theories, e.g., mathlib.

More speculatively:

- Higher directed type theory.
- The type theoretic multiverse, building on the topos theoretic multiverse [BO23] (cf. the set theoretic multiverse [Ham12]) – towards a modal type theory of toposes (cf. modal logic of forcing [HL08]), allowing us to freely move between external and internalization.
- Realizability models of HoTT. Partial work on the effective $(2,1)$-topos [AE25].
- Computability for higher algebraic structures using oracle modalities [Swa24].
- Inner models of HoTT, some progress on constructive L [MR24].
- Homotopical model theory?

Thank you

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