On symmetries of spheres in univalent foundations

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Introduction & Motivation

We study the automorphism groups of the spheres $\mathbb{S}^n$ in homotopy type theory, i.e., the types

$$\mathrm{G}(n+1) :\equiv (\mathbb{S}^n = \mathbb{S}^n)$$

for $n \geq -1$.

The first cases are trivial:

$$(\emptyset = \emptyset) \text{ is contractible}$$
$$(\mathbb{S}^0 = \mathbb{S}^0) = \{\mathrm{id}, \mathrm{swap}\} = \mathbb{S}^0$$

The delooping $\mathrm{BG}(n+1)$ classifies fibrations with fiber $\mathbb{S}^n$ (spherical fibrations).

Aside from the intrinsic interest, this gives a good opportunity to exercise our synthetic homotopy theory muscles.

See also the Symmetry book [BBCDG24].

Symmetries of the circle

Theorem

There is an equivalence

$$(\mathbb{S}^1 = \mathbb{S}^1) \simeq \{\pm 1\} \times \mathbb{S}^1$$

induced by checking orientation preservation/reversal and evaluation at the base point.

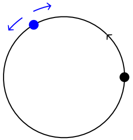
A direct proof is not too hard; it also follows from:

Theorem (after Gottlieb [Got65])

For any group G there is a fiber sequence:

$$\mathrm{BZ}(G) \xrightarrow{\iota}_* (BG = BG) \xrightarrow{p}_* \mathrm{Out}(G)$$

where $(BG = BG)$ is pointed at id_{BG}.



This follows from the fiber sequence

$$(A = A)_{(\mathrm{id}_A)} \xrightarrow{\iota}_* (A = A) \xrightarrow{|-|_0}_* \|A = A\|_0$$

after giving $BZ(G) \simeq_* (BG = BG)_{(\mathrm{id}_{BG})}$ and $\|BG = BG\|_0 \simeq_* \mathrm{Out}\, G$.

Use $\mathbb{S}^1 = \mathrm{B}\mathbb{Z}$ and note that $\mathrm{Z}(\mathbb{Z}) = \mathbb{Z}$ and $\mathrm{Out}(\mathbb{Z}) = \{\pm 1\}$, and that we have a section $s : \{\pm 1\} \to (\mathbb{S}^1 = \mathbb{S}^1)$ of p.

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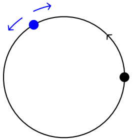
A direct proof is not too hard; it also follows from:

Theorem (after Gottlieb [Got65])

Taking $G = \mathbb{Z}$ there is a fiber sequence:

$$\mathbb{S}^1 \xrightarrow{\iota}_* (\mathbb{S}^1 = \mathbb{S}^1) \xrightarrow{p}_* \{\pm 1\}$$

where $(\mathbb{S}^1 = \mathbb{S}^1)$ is pointed at $\mathrm{id}_{\mathbb{S}^1}$.



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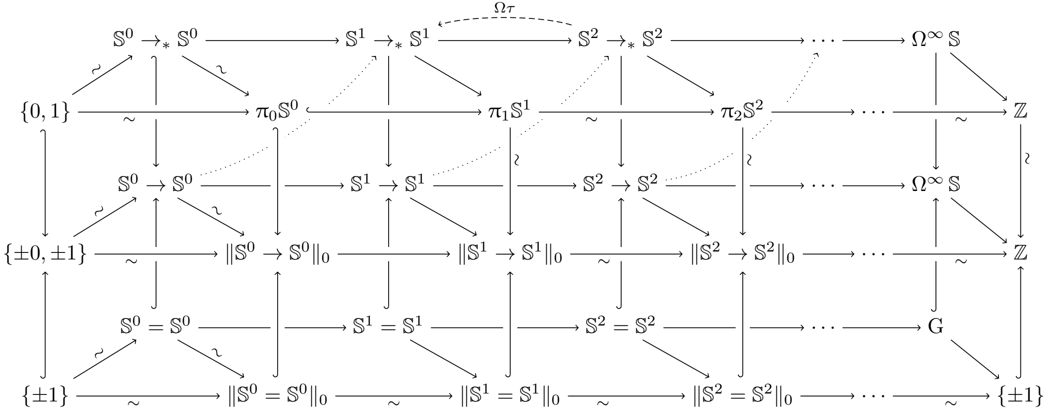
Symmetries of higher spheres and degrees

From Freudenthal and the Hopf retraction $\tau : \Omega\mathbb{S}^2 \to_* \mathbb{S}^1$ of $\eta : \mathbb{S}^1 \to_* \Omega\mathbb{S}^2$ we get:



The colimit of $\mathbb{S}^n = \mathbb{S}^n$ is identified with the symmetries of the sphere spectrum, $\mathrm{G} := (\mathbb{S} = \mathbb{S})$.

Interlude on Whitehead products

The generalized Whitehead product of $\alpha : \Sigma A \to_* X$ and $\beta : \Sigma B \to_* X$ is the composition

$$[\alpha, \beta] :\equiv (\alpha \vee \beta) \circ W_{A,B} : A * B \to_* X,$$

where $W_{A,B} : A * B \to_* \Sigma A \vee \Sigma B$ is adjoint to a "commutator" $A \wedge B \to_* \Omega(\Sigma A \vee \Sigma B)$. Taking A and B to be spheres recovers the classical Whitehead product

$$[-,-] : \pi_{p+1}(X) \times \pi_{q+1}(X) \to \pi_{p+q+1}(X).$$

Fix $\beta : \Sigma B \to_* X$, and consider the fiber sequence of evaluation:

$$(\Sigma B \to_* X)_{(\beta)} \xrightarrow{\iota}_* (\Sigma B \to X)_{(\beta)} \xrightarrow{\mathrm{ev}_\beta}_* X.$$

Theorem (after Lang [Lan73])

For any $\beta : \mathbb{S}^{q+1} \to_* X$, there is a long exact sequence

$$\cdots \longrightarrow \pi_{n+1}(\mathbb{S}^{q+1} \to X, \beta) \to \pi_{n+1}(X) \xrightarrow{\partial_\beta^{n,q}} \pi_{n+q+1}(X) \longrightarrow \pi_n(\mathbb{S}^{q+1} \to X, \beta) \longrightarrow \cdots$$

where the connecting homomorphisms are Whitehead products $\partial_\beta^{n,q} = [-,\beta]$.

Fundamental groups of $(\mathbb{S}^n = \mathbb{S}^n)$

Proposition

For all $n \geq 1$, the two connected components of $(\mathbb{S}^n = \mathbb{S}^n)$ are equivalent.

Note also that $(\mathbb{S}^n \to_* \mathbb{S}^n)$ is a group for $n \geq 1$, so all its components are equivalent.

Theorem

We have $\pi_1(\mathbb{S}^2 = \mathbb{S}^2, \mathrm{id}) = \mathbb{Z}/2\mathbb{Z}$.

We use our LES with $X \equiv \mathbb{S}^2$, $n = q = 1$ and $\beta \equiv \mathrm{id}_{\mathbb{S}^2}$ to get:

$$\pi_2(\mathbb{S}^2) \stackrel{[-,i_2]}{\to} \pi_3(\mathbb{S}^2) \stackrel{\kappa}{\to} \pi_1(\mathbb{S}^2 \to \mathbb{S}^2, \mathrm{id}_{\mathbb{S}^2}) \to 0$$

Thus, $\pi_1(\mathbb{S}^2 = \mathbb{S}^2, \mathrm{id}) = \pi_3(\mathbb{S}^2)/\langle[i_2, i_2]\rangle = \mathbb{Z}/2\mathbb{Z}$ by Brunerie [Bru16].

Indeed, for any $p : a =_A a$, multiplying by p gives an equivalence

$$(a = a)_{(\mathrm{id}_A)} \simeq (a = a)_{(p)}.$$

The case $n \geq 3$ is much easier:

$$\pi_1(\mathbb{S}^n \to \mathbb{S}^n, f) = \mathbb{Z}/2\mathbb{Z}$$

for any $f : \mathbb{S}^n \to \mathbb{S}^n$. Use the LES of the fiber sequence

$$(\mathbb{S}^n \to_* \mathbb{S}^n)_{(f)} \to_* (\mathbb{S}^n \to \mathbb{S}^n)_{(f)} \to_* \mathbb{S}^n$$

and use $\pi_{n+1}(\mathbb{S}^n) = \mathbb{Z}/2\mathbb{Z}$.

Conclusions and future directions

Results:

- We showed that $(\mathbb{S}^n = \mathbb{S}^n)$ has two components, and these have fundamental group $\mathbb{Z}/2\mathbb{Z}$ for $n \geq 2$.
- We developed a version of G.W. Whitehead's EHP long exact sequence.
- Key calculations formalized with cubical agda.

Future directions:

- Classical theorem [Han83]: $(\mathbb{S}^2 = \mathbb{S}^2)_{(\mathrm{id}_{\mathbb{S}^2})} \simeq \mathrm{SO}(3) \times \tilde{\Omega}$, where $\tilde{\Omega}$ is the universal cover of $(\mathbb{S}^2 \to_* \mathbb{S}^2)$ at $\mathrm{id}_{\mathbb{S}^2}$.
- With additivity of Whitehead products, we get $\pi_1(\mathbb{S}^2 \to \mathbb{S}^2, f) = \mathbb{Z}/2\mathrm{deg}(f)\mathbb{Z}$ for $f : \mathbb{S}^2 \to \mathbb{S}^2$.
- EHP spectral sequence ($\rightsquigarrow$ p-primary $\pi_n(\mathbb{S})$).
- Toda brackets (odd primes).

Thank you

References I

[BBCDG24] Marc Bezem, Ulrik Buchholtz, Pierre Cagne, Bjørn Ian Dundas, and Daniel R. Grayson. *Symmetry*. https://github.com/UniMath/SymmetryBook. Commit: c3f12a7. 2024 (cit. on p. 2).

[Bru16] Guillaume Brunerie. "On the homotopy groups of spheres in homotopy type theory". PhD thesis. Université de Nice Sophia Antipolis, June 2016. arXiv: 1606.05916 (cit. on p. 7).

[Got65] Daniel H. Gottlieb. "A Certain Subgroup of the Fundamental Group". In: *American Journal of Mathematics* 87.4 (1965), pp. 840–856. DOI: 10.2307/2373248 (cit. on pp. 3, 4).

[Han83] Vagn Lundsgaard Hansen. "The homotopy groups of a space of maps between oriented closed surfaces". In: *Bull. London Math. Soc.* 15.4 (1983), pp. 360–364. DOI: 10.1112/blms/15.4.360 (cit. on p. 8).

[Lan73] George E. Lang Jr. "The evaluation map and EHP sequences". In: *Pacific J. Math.* 44 (1973), pp. 201–210. URL: http://projecteuclid.org/euclid.pjm/1102948664 (cit. on p. 6).