The $(\infty, 1)$-category of $(\infty, 1)$-categories in simplicial type theory

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Introduction & Motivation

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- In *Univalent Foundations*, done in Homotopy Type Theory, every type of mathematical object is represented by a type A, such that the identity type $a =_A b$ represents the natural notion of *identification* of objects $a, b : A$. (SIP: *Structure Identity Principle*)
- Since every construction respects identity types, this makes all constructions invariant under identification/equivalence.
- E.g., we have 2-type $1\mathrm{Cat}$, such that $C =_{1\mathrm{Cat}} D$ is equivalent to the type of equivalences of categories $C \simeq D$.

However:

- We want the homotopical analog of 1-categories, $(\infty, 1)$-categories, where hom-*sets* are replaced by hom-*types*.
- It's unknown how to define the type Cat, of $(\infty, 1)$-categories, in HoTT following the SIP, and such that types & functions, spectra & maps, etc. are elements.
- This is the famous open problem of (semi-)simplicial types or infinitely coherent objects.
- Plus, an analytic solution might be cumbersome to work with ...

Our Contribution at a Glance

Setting:

- *Simplicial Type Theory* (STT) gives a *synthetic* approach to category theory: augment HoTT with a directed interval $\mathbb{I}$ (plus axioms).
- Intended models: Simplicial objects of a model of HoTT, with simplicial spaces as the classical case.
- All types have arrows $\mathbb{I} \to X$; the *categories* are those with uniquely composable arrows and satisfying a univalence condition (identity $=$ isomorphism).

Our contributions:

- We define Cat as the classifier for *cocartesian fibrations*, cf. the Grothendieck correspondence (straightening–unstraightening):

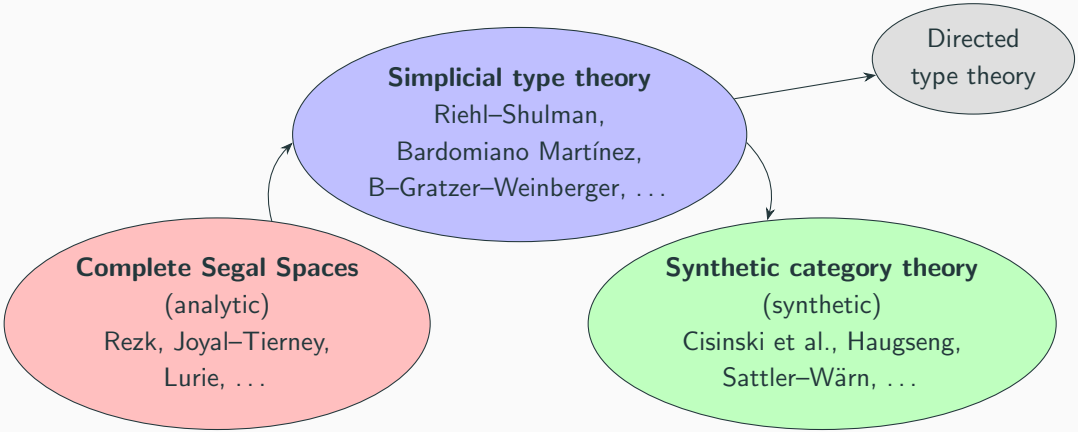
$$\langle \flat \mid X \to \mathsf{Cat} \rangle \to \left\langle \flat \mid \textstyle\sum_{A:X\to\mathcal{U}} \mathsf{isCocart}(A) \right\rangle$$

- This satisfies *directed univalence*:

$$\hom_{\mathsf{Cat}}(A, B) \simeq \langle \flat \mid A \to B \rangle$$

- This generalizes the SIP to the SHP: the *Structure Homomorphism Principle*.

Simplicial Type Theory as a Bridge



Simplicial type theory functions as a bridge, working with simplicial spaces synthetically to make the analytic theory of higher categories easier.

Simplicial Type Theory

Simplicial Type Theory

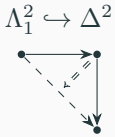
In simplicial type theory [RS17] we interpret HoTT in simplicial spaces, the classifying topos of an *interval* $\mathbb{I}$ (a totally ordered set with distinct 0 and 1).

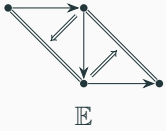
From $\mathbb{I}$ we build simplices $\Delta^n := \{\, x_1, \ldots, x_n : \mathbb{I} \mid x_1 \geq \cdots \geq x_n \,\}$.

Definition X is *Segal* if $X^{\Delta^2} \to X^{\Lambda^2_1}$ is an equivalence.

Definition A Segal type X is *Rezk* if $X^{\mathbb{E}} \to X$ is an equivalence, for $\mathbb{E}$ the "walking equivalence".

The Rezk types are categories; Segal types are precategories. It's possible to do quite a lot of abstract category theory with these, incl. fibered category theory [BW23].





Modal Triangulated Type Theory

Modal Triangulated Type Theory

But there aren't many *examples* of categories in bare simplicial type theory. In previous work we constructed the category of spaces $\mathcal{S}$ [GWB24] and proved the Yoneda lemma [GWB25] (LICS '25). Here we close one of the remaining problems and construct Cat.

As for $\mathcal{S}$, we define Cat using the [LOPS18] construction, so we need $\mathbb{I}$ to be *tiny*[1], which it isn't in simplicial spaces; thus we work in cubical spaces: presheaves on the full subcategory on $\{(0 < 1)^n\}$ of Cat.

With MTT [GKNB21; GKNB20], we add modalities:

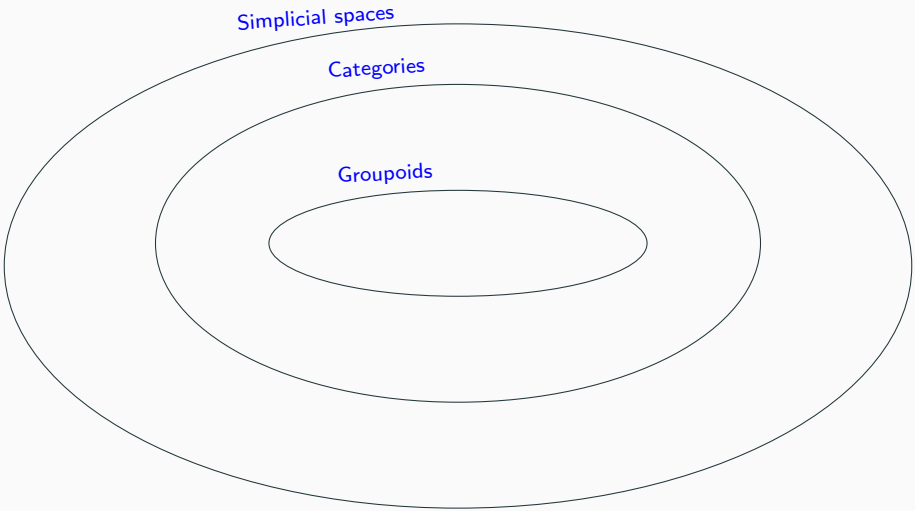
- idempotent comonad $\flat$ (global/objectwise) with right adjoint $\sharp$,
- involutive op (opposite),
- tw (twisted arrows) with 2-cells $\mathrm{tw} \to \mathrm{id}$, $\mathrm{tw} \to \mathrm{op}$.

Axioms include: $\mathbb{I}$ is a tiny bounded distributive lattice with $\langle \flat \mid 2 \rangle \simeq \langle \flat \mid \mathbb{I} \rangle$ and de Morgan duality $\langle \mathrm{op} \mid \mathbb{I} \rangle \simeq \mathbb{I}$.

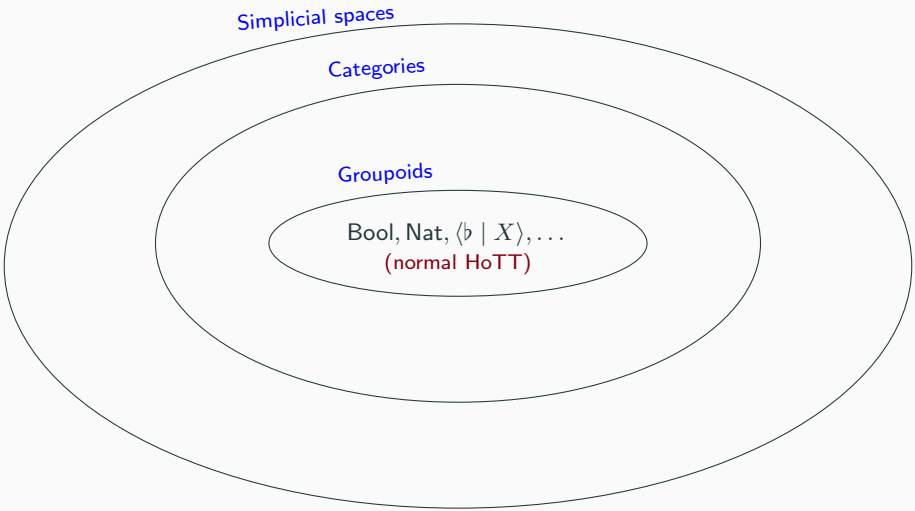
[1] $\mathbb{I}$ is tiny if $\mathbb{I} \to _$ has a right adjoint.

See also Riley [Ril24], and Weaver–Licata [WL20].

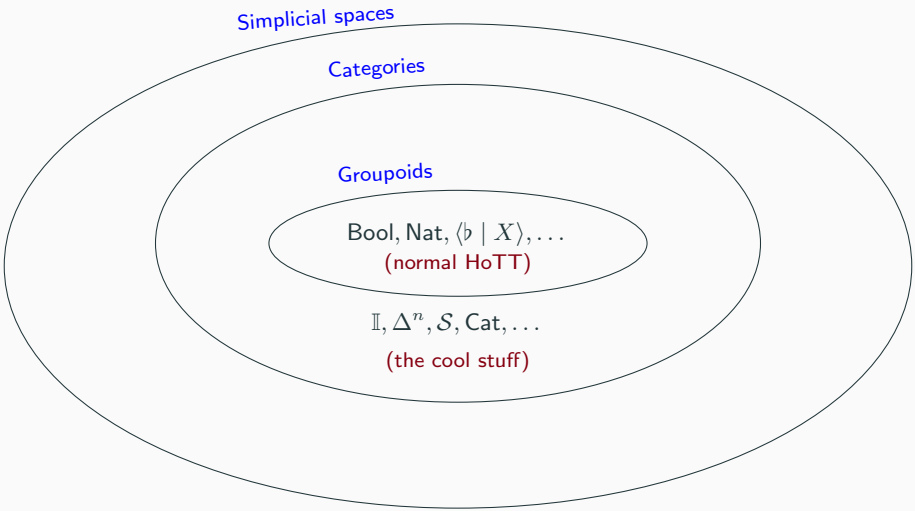
A Map of $\mathbf{PSh}(\Delta)$



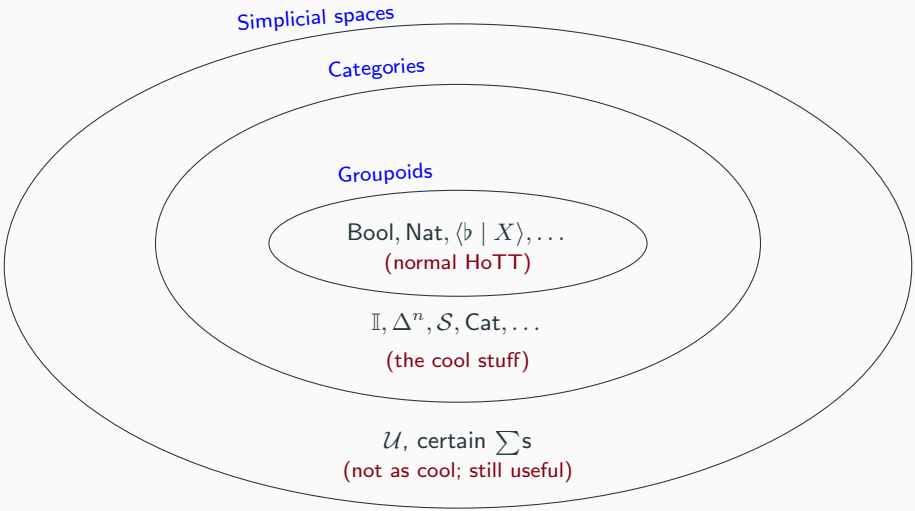
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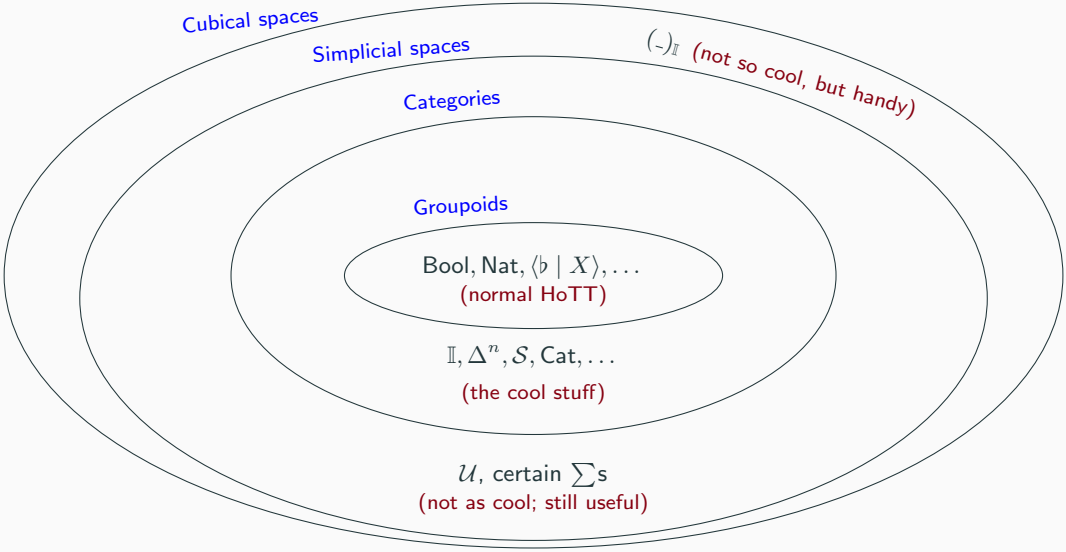
A Map of $\mathbf{PSh}(\Delta)$



A Map of $\mathrm{PSh}(\Delta)$



A Map of $\mathbf{PSh}(\square)$



Triangulated Type Theory

Cubical spaces contain simplicial spaces as a subtopos [SW21], which we can define as the lex modality $\diagup\!\!\diagup$, nullification at the family of propositions:

$$\Phi : \mathbb{I} \times \mathbb{I} \to \mathsf{HProp}, \qquad \Phi(i,j) = (i \leq j) \vee (j \leq i)$$

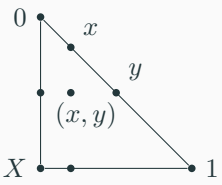
The $\diagup\!\!\diagup$-modal types are the simplicial spaces. We add as axioms that cubes are projective generators [MR23].

Further, we use synthetic quasicoherence [Ble23], which relies on the fact that $\mathbb{I}$, qua cubical space, is simplicial, see also [Spi16].

Define an $\mathbb{I}$-algebra to be a bounded distributive lattice R under $\mathbb{I}$.

Axiom (Duality) For any finitely presented $\mathbb{I}$-algebra R, the map $R \to (\hom_{\mathbb{I}}(R, \mathbb{I}) \to \mathbb{I})$ is an equivalence.





Cor. (Phoa's principle) We have equivalences

$$\Delta^2 \simeq \mathbb{I}[X] \simeq (\mathbb{I} \to \mathbb{I}).$$

Recap: The Category of Spaces

(Amazingly) Covariant Families and the Category of Spaces

Consider the notion of covariant families:

$$\mathsf{isCov} : \mathcal{U}^X \to \mathsf{HProp}$$

capturing families right orthogonal to $\{0\} \to \mathbb{I}$.

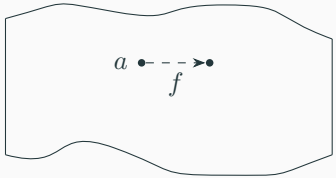
Theorem $\mathcal{S} \coloneqq \mathcal{U}_{\mathsf{ACov}} \coloneqq \mathcal{U} \times_{\mathsf{HProp}_{\mathbb{I}}} \mathbf{1}_{\mathbb{I}}$ classifies amazingly covariant families: for all $X :_\flat \mathcal{U}$, the canonical map

$$\langle \flat \mid X \to \mathcal{S} \rangle \to \left\langle \flat \mid \textstyle\sum_{A : X \to \mathcal{U}} \mathsf{isCov}(A) \right\rangle$$

is an equivalence.

Theorem $\mathcal{S}$ is a category satisfying directed univalence: $\hom_{\mathcal{S}}(A, B) \to (A \to B)$ is an equivalence.

To get simpliciality, we axiomatize that $\triangleleft A$ has the same n-simplices as A.





Now: The Category of Categories

(Amazingly) Cocartesian Families and the Category of Categories

Consider the notion of cocartesian families:

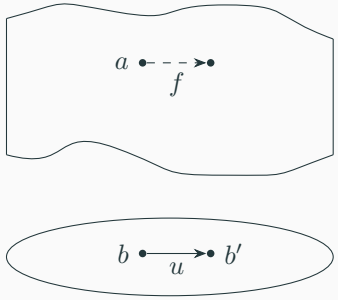
$$\mathrm{isCocart} : \mathcal{U}^X \to \mathsf{HProp}$$

capturing families of categories with *initial sections*: Given $u : \mathbb{I} \to X$ and $a : A(u\,0)$, the type of lifts $\sum_{f:(i:\mathbb{I})\to A(u\,i)} f\,0 = a$ has an initial element.
(See Schellingerhout [Sch26])

Theorem $\mathsf{Cat} := \sum_{A:\mathcal{U}_{\mathbb{S}}} \mathsf{isRezk}\,A \times \mathsf{isInner}\,A \times \mathsf{aHasNLCCLifts}\,A$ is the base of the universal cocartesian family, i.e., for all $X :_\flat \mathcal{U}$, the canonical map

$$\langle \flat \mid X \to \mathsf{Cat} \rangle \to \left\langle \flat \mid \textstyle\sum_{A:X\to\mathcal{U}} \mathsf{isCocart}(A) \right\rangle$$

is an equivalence.





Directed Univalence Reloaded and Straightening–Unstraigtening

Directed univalence If A, B are $\flat$-elements of Cat, then there is a equivalence
$\mathsf{dua} : \hom_{\mathsf{Cat}}(A, B) \simeq \langle \flat \mid A \to B \rangle$.

More generally, we get an equivalence between $\langle \flat \mid X \times \mathbb{I} \to \mathsf{Cat} \rangle$ and $\left\langle \flat \mid \sum_{A_0, A_1 : \mathsf{Cat}^X} A_0 \to^{\mathsf{cc}} A_1 \right\rangle$.

With a bit more work, we characterize cocartesian families $A :_\flat X \times \Delta^2 \to \mathcal{U}$, allowing us to prove that Cat is Segal and Rezk. The hardest part is then to show Cat is simplicial, which can be done by giving a suitable map back $\mathbb{\Delta}\mathsf{Cat} \to \mathsf{Cat}$.

As for $\mathcal{S}$, directed univalence is implemented via *directed gluing*, mapping a cocartesian functor $\alpha : \prod_{x:X} F_0\, x \to F_1\, x$ to
$\mathsf{Gl}(F_0, F_1, \alpha) : X \times \mathbb{I} \to \mathcal{U}_{\mathbb{\Delta}}$
$\mathsf{Gl}(F_0, F_1, \alpha)(x, i) = \sum_{y : F_1(x)} (\alpha(x)^{-1}(y))^{i=0}$

Full Straightening–Unstraightening We get Lurie's straightening–unstraightening theorem by adapting [CCNW25]. The unstraightening map

$$(C \to \mathsf{Cat}) \to \mathsf{Cat}/C$$

is an embedding whose image is the subcategory $\mathsf{Cocart}(C)$ consisting of cocartesian functors and cocartesian transformations.

Applications and Outlook

Examples

- The map $i : \mathcal{S} \to \mathrm{Cat}$ is fully faithful and possesses both left and right adjoints.
- The inclusion $\mathrm{Cat}_n \hookrightarrow \mathrm{Cat}$ has a left adjoint.
- The category of monoidal categories MCat is the full subcategory of $\langle \mathrm{op} \mid \Delta \rangle \to \mathrm{Cat}$ spanned by monoidal categories, those $C^{\otimes} : \mathrm{Cat}^{\langle \mathrm{op} \mid \Delta \rangle}$ for which $(\rho_n^1, \dots, \rho_n^n) : C^{\otimes}([n]) \to C^{\otimes}([1]) \times \cdots \times C^{\otimes}([1])$ is an equivalence for all n.
- The category of E_1-monoids is similarly a subcategory of $\mathcal{S}^{\langle \mathrm{op} \mid \Delta \rangle}$.
- We get a simple form of algebraic K-theory as a composite:

$$k : \mathrm{MCat} \to \mathrm{Mon} \to \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*$$

  (Use that $(-)^{\simeq} : \mathrm{Cat} \to \mathcal{S}$ is a right adjoint, inducing a functor MCat $\to$ Mon.)
- $\sum_{A:\mathrm{Cat}} A^C$ is the lax slice category.
- $\mathrm{Cat}_{\mathrm{marked}} = \sum_{C:\mathrm{Cat}} ((\mathbb{I} \to C)^{\simeq} \to \mathbb{I})$ is the category of marked categories.
- More examples of the SHP to follow ...

Formalization Projects

Bespoke proof assistant `Rzk` by Nikolai Kudasov [Kud23]:

- Main feature: Extension types $\left\langle \prod_{t:\Psi} P(t) \middle|_a^{\Phi} \right\rangle$, incl. strict hom-types $\hom_A(a, a') \coloneqq \left\langle \Delta^1 \to A \middle|_{[a,a']}^{\partial\Delta^1} \right\rangle$.
- Good tool integration in VS Code.
- Formalization of the [RS17] Yoneda lemma [KRW04].
- Recently extended with MTT, towards a formaliation of $\mathcal{S}$ [TK26]

The agda-synthetic-categories project by Sam Toth, Fredrik Bakke, Mark Williams, et al.

- Work in standard Agda, now extended with MTT for $\flat$ and $\sharp$ [Tot26].
- Work (harder) with reflective subuniverses/modalities [RSS20] and the Leibniz adjunction [JKL26] to reason with homotopy extensions.
- Lifting conditions (spine and necklace, ...); associativity + pentagon law.
- Construction & directed univalence of $\mathcal{S}$.

Outlook

Some next steps:

- Exponentiability of (co)cartesian functors (WIP)
- Higher algebra & operads
- Toposes and models of type theory
- A computational account, building on [WL20]
- Constructive models in all higher toposes, building on [CHS26]
- Formalize everything

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To (infinity, infinity) and beyond!

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To (infinity, infinity) and beyond!

Thank you




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