Of Cubes and Spaces With and Without the Axiom of Choice

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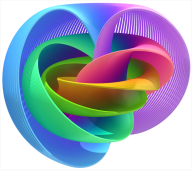
Ljubljana, 12 November 2026



Braque's *Maisons à l'Estaque*

Homotopy Theory & Type Theory

Homotopy deals with maps *up to* homotopy (e.g., $f : S^3 \to S^2$) and spaces *up to* homotopy equivalence.
Gradually, we came to view the resulting entities as a new kind of object (∞-groupoids, *anima*) and we see them popping up all over mathematics (cf. Manin's *homotopical picture of the world*).



Martin-Löf type theory is a theory of mathematical constructions ($\sum$, $\prod$, $=_A$, $+$, $\mathbb{N}$, $\mathcal{U}_i$, $\dots$), done in *contexts* Γ of typed variables $x : A$.

For a while we didn't know whether any two proofs of equality $p, q : a =_A b$ could be proved equal. But we found the *groupoid* and then ∞-*groupoid* models, and thus was born the field of *homotopy type theory*.

(Homotopy Type) Theory = Homotopy (Type Theory)

Cubical Type Theory

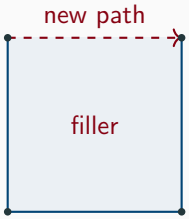
Univalence and higher inductive types imposed as axioms destroy the computational nature of type theory. (Though we still have *homotopy canonicity*.) To get a better type theory, we need *new constructions*.

Cubical type theory adds interval variables $i : \mathbb{I}$ and interprets paths as maps

$$p : \mathbb{I} \to A, \qquad p(0) = a, \quad p(1) = b.$$

A filling operation extends partial cubes, commuting with substitution in remaining variables.

Cohen et al. (2018) and Coquand et al. (2018b); cofibration axioms: Orton and Pitts (2018, §3).



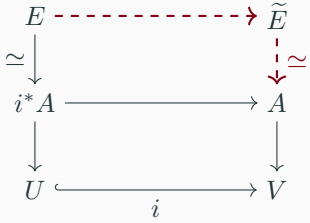
given: three sides
produced: a square

Type-Theoretic Model Structures

- **Cofibrations**: levelwise decidable inclusions (all monomorphisms classically).
- **Fibrations**: families with uniform box fillers.
- **Weak equivalences**: maps factoring as a trivial cofibration followed by trivial fibration.

Sattler's equivalence extension property supplies the model-structure machinery.

Sattler (2017, Thm. 8.1, Cor. 8.2); Awodey (2023) and Cavallo, Mörtberg, et al. (2020).



extend a fibrant family and its equivalence from U to V

Varieties of Cubical Sets

A cubical set is a presheaf $X : \square^{\mathrm{op}} \to \mathbf{Set}$. The choice of allowed substitutions matters.

Cube category	Substitutions / extra operations	Example
BCH	faces, projections, permutations	no repeated variable
Cartesian	also diagonals	$(i, j) \mapsto (i, i)$
One connection	cartesian $+ \vee$	$(i, j) \mapsto i \vee j$
Dedekind	cartesian $+ \wedge, \vee$; lattice laws	all monotone Boolean maps
De Morgan (CCHM)	also $\neg$; De Morgan laws	$i \mapsto \neg i$

Being a test category does not by itself identify the *type-theoretic* model structure.

Selected rows, adapted from Buchholtz and Morehouse (2017); Bezem et al. (2014) and Cohen et al. (2018).

Early Counterexamples

Two independent arguments (2018): familiar cubical models need not give the familiar homotopy theory.

Sattler: for ordinary cartesian cubes,

$$\mathbb{I}^2/\Sigma_2 \not\simeq 1$$

in the uniform type-theoretic model, although it is contractible in the test model.

B: similar example for cubes with reversals. The category of elements yield $\mathbb{RP}^\infty$, rather than a naive geometric quotient.



coordinate swap

De Morgan: $X = \mathbb{I}/(i \sim \neg i)$

$$BC_2 \xrightarrow{\text{right cofinal}} \square/X$$

$$|N(\square/X)| \simeq \mathbb{RP}^\infty$$

Coquand et al. (2018a). Cartesian argument: Coquand (2018) and Sattler (2018); Awodey et al. (2026, §1.5.1).

Positive Results

Model	Key idea	Reference
One connection	Relative elegance; classical comparison with spaces	Cavallo–Sattler (preprint 2022/23; journal 2025)
Equivariant cartesian	Fill compatibly with coordinate symmetries; classical comparison	Awodey–Cavallo–Coquand–Riehl–Sattler (2026)
(Semi)simplicial	Constructive Kan–Quillen theory and univalent models	Gambino–Henry (2022); Gambino–Sattler–Szumiło (2022); Henry (2020)

Dedekind cubes resisted the available methods.

Cavallo and Sattler (2025, Thms. 7.6–7.8, App. A.2); Awodey et al. (2026, Thm. 6.3.6); Gambino and Henry (2022), Gambino, Sattler, et al. (2022), and Henry (2020).

The Dedekind Cubes

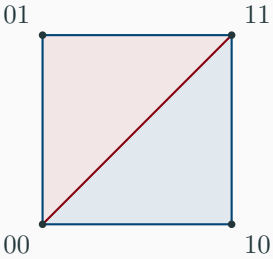
$$B_n = \{0 < 1\}^n, \qquad D(B_m, B_n) = \mathbf{Pos}(B_m, B_n).$$

Every coordinate is a monotone Boolean function, hence built from $0, 1, \wedge, \vee$.

The representable interval is $\mathbb{I} = yB_1$. Its generalized points are not just 0 and 1!

Chains are retracts of cubes. For their inclusion $j : \Delta \to \mathrm{Kar}(D)$, restriction gives triangulation j^*.

Sattler (2019); Streicher and Weinberger (2021, §3).



two maximal chains in B_2

Sattler's Comparison

Use chains *without degeneracies*:
$m : \Delta_+ \to \mathrm{Kar}(D)$.

$$m_! \dashv m^* \dashv m_*, \qquad M = m_* m^*.$$

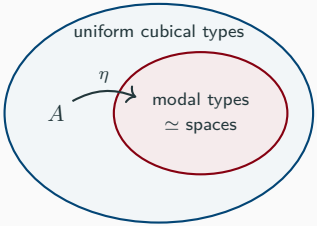
On fibrant objects, M is homotopy idempotent:

$$MA \simeq MMA.$$

Its modal objects present spaces constructively.
With degeneracies, $s = j_* j^*$ is strictly idempotent, but $j^* \dashv j_*$ is not constructively Quillen.

Must $\eta_A : A \to MA$ already be an equivalence?

Sattler (2025, slides 29–37); Henry (2020, Thm. 5.5.6).



Why use M constructively?

- No degeneracy tests: all semisimplicial sets are cofibrant.
- Fillers assemble uniformly, so M preserves fibrant structure.

Searching for a Counterexample

Originally I thought there would be non-good cubical sets. So I (vibe-)coded a search among subquotients of cubes, in Python then Rust, using contractions, covers, and "supported" quotients to certify the good ones.

The number of maps $B_k \to B_n$ is d_k^n, where d_k is the k^{th} *Dedekind number*.

At level 4, even $\mathbb{I}^3$ has

$$168^3 = 4\,741\,632 \text{ cubes.}$$

These finite certificates suggested a proof.

k	d_k	d_k^3
0	2	8
1	3	27
2	6	216
3	20	8 000
4	168	4 741 632
5	7 581	435 691 903 941

What Would Count as a Proof?

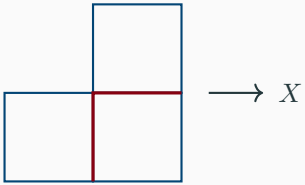
Call X *good* if it is correctly assembled from its cubes:

$$\operatorname*{hocolim}_{(B_n,x)\in D/X} \mathbb{I}^n \xrightarrow{\simeq} X.$$

Classically, good objects contain the strictly contractible ones and are closed under:

- pushouts along monomorphisms;
- increasing transfinite unions;
- retracts.

Cisinski (2006, Prop. 3.4.22).



pieces *and their overlaps*

Good does not mean contractible.

A New Induction: Count Target Vertices

For a finite poset Q, set

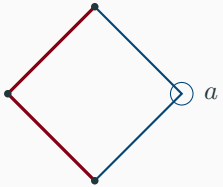
$$N_D(Q)(B_n) = \mathbf{Pos}(B_n, Q).$$

Prove *every* S/E good, for $S \subseteq N_D(Q)$ and every equivalence relation E.

Induct on $|Q|$. A cube is full if it hits every vertex. The non-full part is

$$T = \bigcup_{a \in Q} S \cap N_D(Q \setminus \{a\}).$$

Each piece, and each overlap, has a smaller target.



a cube missing a
factors through $Q \setminus \{a\}$

Inductive payoff: every quotient of T, and every subobject of such a quotient, is good.

If $S = T$, we are done.

Full Cubes Have a Common Bottom

If $u : B_m \twoheadrightarrow Q$ is full, then $u(0) = b = \min Q$.

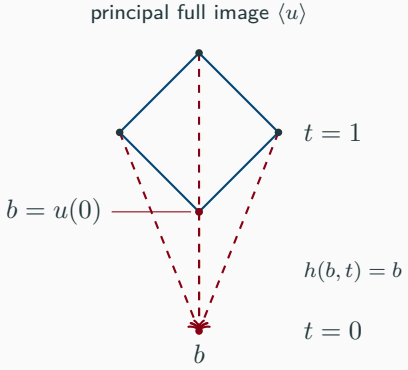
The monotone map

$$h : Q \times B_1 \longrightarrow Q,$$
$$h(z, 0) = b, \qquad h(z, 1) = z$$

induces $h : N_D(Q) \times \mathbb{I} \to N_D(Q)$. We need the slide identity:

$$\boxed{h(ug, t) = u(g \wedge t)}$$

Put $U = \bigcup_{u \text{ full}} \langle u \rangle$. Then h contracts U to b and respects every *full/full* relation.



A Constructive Lemma: A Quotient Supported on a Subobject

For a set W, let $p : W \to X = W/\rho$ be a set quotient, $B \subseteq X$, and $A = p^{-1}(B)$.

Suppose every generating identification is trivial or lies over B (so B supports the quotient):

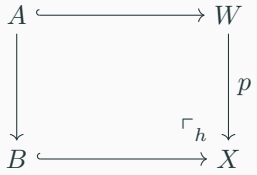
$$\rho(w, w') \implies (w = w') \vee B(pw)$$

Quotient elimination gives

$$\left\| W \amalg^h_A B \right\|_0 \simeq X.$$

If A, W, B are sets and $A \hookrightarrow W$ is an **embedding**, the homotopy pushout is already a set:

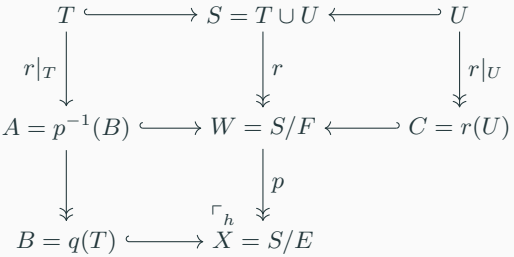
$$\boxed{X \simeq W \amalg^h_A B.}$$



Cubical application: $A \hookrightarrow W$ is a cofibration, so the strict pushout is also a homotopy pushout.

Finish the Induction: The Remaining Relations Meet T

Let $F \subseteq E$ be generated by full/full pairs; $q = pr$. Then p is supported on $B = q(T)$.



$D = r(T)$ and $B = q(T)$, with all subobjects: good by induction.

$C = r(U)$: contractible.

$K \subseteq C$: generated by images of full cubes E-related to non-full ones; empty or contractible. Then

$$W = D \cup C, \qquad A = D \cup K,$$

with both overlaps subobjects of D.

Hence W, A, B are good, and the supported-quotient square gives

$$\boxed{X \cong W \amalg_A B \text{ is good.}}$$

First Nontrivial Case: Do Not Contract the Circle!

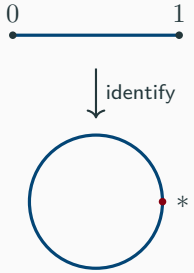
Take $Q = B_1$, $S = \mathbb{I}$, and identify its endpoints.

This relation is *non-full/non-full*, so it survives until the final pushout.

$$\mathbb{I} \amalg_{\partial \mathbb{I}} 1 \simeq S^1.$$

The interval contracts, but that contraction does not respect the endpoint identification.

(And $\{0, 1\} \to \mathbb{I}$ isn't a homotopy embedding, which fits, since S^1 isn't a set.)



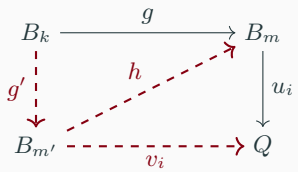
Why the Finite Part Is Constructive

A *locally finite* cubical set is one with finitely many n-cubes for each n.

Bounding Lemma For full $u_1, \ldots, u_r : B_m \to Q$ any $g : B_k \to B_m$ factors through some $B_{m'}$, $g = hg'$, with bound m' is independent of m, and each $v_i = hu_i$ still full.

1. Keep $\operatorname{im} g$ and preimages witnessing fullness: at most $n = 2^k + r|Q|$ vertices.
2. The generated lattice $L \subseteq B_m$ is a quotient of the free bounded distributive lattice on n generators: $|L| \leq d_n$.
3. Birkhoff duality ($L \simeq \mathrm{Down}(J(L))$) gives a monotone retraction $B_{m'} \twoheadrightarrow L$, with

$$\boxed{m' = |J(L)| \leq |L| \leq d_{\,2^k + r|Q|}.}$$



d_n: Dedekind number
$J(L)$: join-irreducibles of L

For **decidable** S, E: $r = 2$ bounds the generators of $F(B_k)$, giving finite equivalence closure at level k.

Thus $W = S/F$ is locally finite and $A = p^{-1}(q(T)) \hookrightarrow W$ is levelwise decidable: a cofibration.

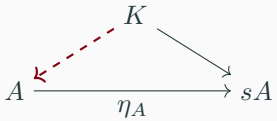
Constructive and Classical Comparison

Constructively

With W-types and universes, for uniformly fibrant A:

$$[K, A] \cong [K, sA] \quad (K \text{ locally finite}),$$

where $s = j_* j^*$ is the simplicial comparison. In particular, all cubical homotopy groups agree.

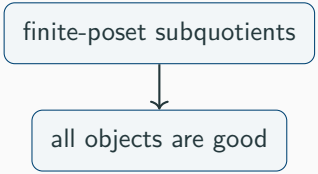


Classically (with choice)

Cisinski's cellular generation promotes the induction to *all* cubical sets.

$$\widehat{D}[W_{\mathrm{tt}}^{-1}] \simeq \mathcal{S}.$$

And the type-theoretic and test model structures coincide.



Classical cellular generation: Cisinski (2006, Prop. 1.2.27, Lem. 1.2.30).

A Countermodel

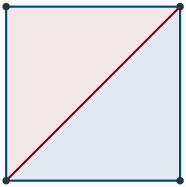
Let $T_2 \subseteq \mathbb{I}^2$ contain precisely the cubes with chain image, and let $R = R(T_2)$ be its free uniformly fibrant replacement.

The inclusion $T_2 \to R$ corresponds to some $a \in sR(B_2)$. A map $\Psi : sR \to R$ natural in algebra maps would give $\sigma = \Psi(a)$ satisfying

$$\boxed{\sigma c_v = e_v \sigma \quad (v \in B_2),}$$

where c_v is the constant substitution, and e_v is induced by the constant endomorphism of T_2.

Free-filler method: Coquand (2018).



T_2 is internally $\sum_{i,j:\mathbb{I}} i \leq j \vee j \leq i$
The identity square is missing

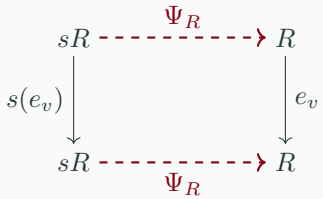
No Term Can Satisfy the Equations

Analyze the filler term $\sigma = \Psi(a)$ in $R = R(T_2)$:

- A **generator** would fix all four vertices: it must be id_{B_2}, which has no chain image.
- A **filler** forces the same equations on its base: descend to a proper subterm.

Thus no such natural map Ψ exists.

The obstruction is *naturality*, and also applies to the one-connection case.



the required commuting square has no solution

From Naturality to Constructive Unprovability

Let $\mathcal{K}$ be the full category of uniform Kan algebras on the two objects $\varnothing$ and R. Work in the diagram topos $\mathbf{Set}^{\mathcal{K}}$.

Its generic diagram G is uniformly fibrant. An internal map $sG \to G$ would give precisely the forbidden natural transformation.

Conclusion. No metatheory valid in these presheaf models proves that every uniformly fibrant A admits a map $sA \to A$. In particular, it cannot prove all the units are equivalences.

The same obstruction applies to M.



internal maps must commute with these endomorphisms

At the initial stage $\varnothing$, existence requires a coherent family.

Auto-Formalization

Following **Licata–Orton–Pitts–Spitters**, work in the extensional internal language of cubical sets.

Strict equality is extensional; homotopy is modeled by $\mathbb{I}$. Crisp variables and $\flat, \sharp$ express external data internally, allowing, e.c., to express choice for discrete sets.

Agda checks the internal classical comparison and the countermodel (relative to a duality principle).

Licata et al. (2018), Shulman (2018, Sharp AC).

Layer	Role
Constructive	finite data, term model
Cubical	interval, fillers, modality
Classical	excluded middle, choice

Checked internal theorems
$+$ semantic interpretation
$\Longrightarrow$ external result

Open Problems

1. Does **excluded middle alone** suffice, without choice?
2. Which **internal HoTT statements** distinguish the cubical model from its modal localization?
3. Can we give a more direct **site description** of the unlocalized constructive model?

Finite probes see spaces, but the full comparison is not constructive: a new kind of fuzzy point.



Braque's *Paysage de l'Estaque*

Math in the AI Age – On Beauty and Truth

Takehome messages:

- We do mathematics for **human flourishing**.
- AI can help us find **truth** – especially when matched with formalization.
- We need to search for **beauty** – which is anyways in the eye of the beholder.



Braque's *Viaduc à l'Estaque*

Thank you!

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