Normalization in Intuitionistic Set Theories

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Outline

Introduction

We summarize the work by Wojciech Moczydłowski, primarily from his 2007 PhD dissertation, Investigation on Sets and Types, supervised by Robert Constable and Richard Shore at Cornell.

The thesis was awarded the 2007 Sacks Prize by the ASL.

Any mistakes in the following are almost surely mine.

IPC

We warm up by studying intuitionistic propositional logic. Formulas:

$$\varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi$$

Rules:

$$\frac{}{\Gamma, \varphi \vdash \varphi} \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi} \qquad \frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi} \qquad \frac{\Gamma \vdash \varphi \to \psi \quad \Gamma \vdash \varphi}{\Gamma \vdash \psi}$$

$$\frac{\Gamma \vdash \varphi \quad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi} \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi} \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi}$$

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi} \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi} \qquad \frac{\Gamma \vdash \varphi \vee \psi \quad \Gamma, \varphi \vdash \theta \quad \Gamma, \psi \vdash \theta}{\Gamma \vdash \theta}$$

The $\lambda^{\to}$ calculus

Simply-typed lambda calculus, $\lambda^{\to}$. The *types* are IPC formulas. The (raw) terms are given by

$$
\begin{aligned}
M ::= x \mid M\,N \mid \lambda x{:}\varphi, M \mid \mathsf{inl}(M) \mid \mathsf{inr}(M) \\
\mid \mathsf{case}\, M\, \mathsf{of} \begin{pmatrix} \mathsf{inl}\, x{:}\varphi \Rightarrow N_1 \\ \mathsf{inr}\, y{:}\psi \Rightarrow N_2 \end{pmatrix} \\
\mid \langle M, N \rangle \mid \mathsf{fst}(M) \mid \mathsf{snd}(M) \mid \mathsf{magic}(M)
\end{aligned}
$$

These correspond exactly to the inference rules for IPC, and give notations for IPC proofs.

Typing for $\lambda^{\to}$

Typing rules for terms of $\lambda^{\to}$ match the rules of IPC:

$$\frac{}{\Gamma, x{:}\varphi \vdash x{:}\varphi} \qquad \frac{\Gamma \vdash M{:}\bot}{\Gamma \vdash \mathsf{magic}(M){:}\varphi}$$

$$\frac{\Gamma, x{:}\varphi \vdash M{:}\psi}{\Gamma \vdash (\lambda x{:}\varphi, M){:}\varphi \to \psi}\; x \notin \mathsf{dom}(\Gamma)$$

$$\frac{\Gamma \vdash M{:}\varphi \to \psi \qquad \Gamma \vdash N{:}\varphi}{\Gamma \vdash (M\ N){:}\psi}$$

$$\frac{\Gamma \vdash M{:}\varphi \qquad \Gamma \vdash N{:}\psi}{\Gamma \vdash \langle M, N\rangle{:}\varphi \wedge \psi}$$

$$\frac{\Gamma \vdash M{:}\varphi \wedge \psi}{\Gamma \vdash \mathsf{fst}(M){:}\varphi} \qquad \frac{\Gamma \vdash M{:}\varphi \wedge \psi}{\Gamma \vdash \mathsf{snd}(M){:}\psi}$$

Typing for $\lambda^{\rightarrow}$, II

Typing rules for disjunction:

$$\frac{\Gamma \vdash M{:}\varphi}{\Gamma \vdash \mathsf{inl}(M){:}\varphi \vee \psi} \qquad \frac{\Gamma \vdash M{:}\psi}{\Gamma \vdash \mathsf{inr}(M){:}\varphi \vee \psi}$$

$$\frac{\Gamma \vdash M{:}\varphi \vee \psi \qquad \Gamma, x{:}\varphi \vdash N_1{:}\theta \qquad \Gamma, y{:}\psi \vdash N_2{:}\theta}{\Gamma \vdash \mathsf{case}\, M\, \mathsf{of} \begin{pmatrix} \mathsf{inl}\, x{:}\varphi \Rightarrow N_1 \\ \mathsf{inr}\, y{:}\psi \Rightarrow N_2 \end{pmatrix}{:}\theta}$$

Note: If $\Gamma \vdash M{:}\varphi$, then $\mathsf{FV}(M) \subset \mathsf{dom}(\Gamma)$.

Reduction for $\lambda^{\rightarrow}$

For our later application to set theory, it essential to use a *deterministic reduction*. Reduce terms to *values*:

$$V ::= \lambda x{:}\varphi, M \mid \mathsf{inl}(M) \mid \mathsf{inr}(M) \mid \langle M, N \rangle$$

Non-values have a principal argument:

In $M\ N$, M is the principal argument.

In $\mathsf{case}\, M\, \mathsf{of} \begin{pmatrix} \mathsf{inl}\, x{:}\varphi \Rightarrow N_1 \\ \mathsf{inr}\, y{:}\psi \Rightarrow N_2 \end{pmatrix}$, M is the principal argument.

In $\mathsf{fst}(M)$, $\mathsf{snd}(M)$ and $\mathsf{magic}(M)$, M is the principal argument.

Reduction for $\lambda^{\to}$, II

A non-value whose principal argument is value may be reduced:

$$
\begin{aligned}
\mathsf{fst}\langle M, N\rangle &\longrightarrow M \\
\mathsf{snd}\langle M, N\rangle &\longrightarrow N \\
(\lambda x{:}\varphi, M)N &\longrightarrow M[N/x] \\
\mathsf{case}\ \mathsf{inl}\, M\ \mathsf{of} \begin{pmatrix} \mathsf{inl}\, x{:}\varphi \Rightarrow N_1 \\ \mathsf{inr}\, y{:}\psi \Rightarrow N_2 \end{pmatrix} &\longrightarrow N_1[M/x] \\
\mathsf{case}\ \mathsf{inr}\, M\ \mathsf{of} \begin{pmatrix} \mathsf{inl}\, x{:}\varphi \Rightarrow N_1 \\ \mathsf{inr}\, y{:}\psi \Rightarrow N_2 \end{pmatrix} &\longrightarrow N_2[M/x]
\end{aligned}
$$

If the principal argument is a non-value, then *that* may be reduced (*lazy* evaluation).

Properties for $\lambda^{\rightarrow}$

Lemma (Correspondence)

If $\Gamma \vdash O{:}\varphi$*, then* $\mathrm{rg}(\Gamma) \vdash \varphi$*. If IPC* $+ \Gamma \vdash \varphi$*, then there is a term* M *of* $\lambda^{\rightarrow}$ *so that* $\Gamma' \vdash M{:}\varphi$*.*

Lemma (Inversion)

We can determine the final typing judgment in a proof by inspecting the proof term.

Lemma (Subject-reduction)

If $\Gamma \vdash M{:}\varphi$ *and* $M \longrightarrow N$*, then* $\Gamma \vdash N{:}\varphi$*.*

Lemma (Progress)

Non-values can always be reduced in an empty context.

Realizability for $\lambda^{\to}$

The terms of the untyped calculus $\overline{\lambda^{\to}}$ are obtained from $\lambda^{\to}$ by erasing the types:

$$\begin{aligned} M ::= {} & x \mid M\ N \mid \lambda x, M \mid \mathsf{inl}(M) \mid \mathsf{inr}(M) \\ & \mid \mathsf{case}\, M\, \mathsf{of} \begin{pmatrix} \mathsf{inl}\, x \Rightarrow N_1 \\ \mathsf{inr}\, y \Rightarrow N_2 \end{pmatrix} \\ & \mid \langle M, N \rangle \mid \mathsf{fst}(M) \mid \mathsf{snd}(M) \mid \mathsf{magic}(M) \end{aligned}$$

We can erase the types of $\lambda^{\to}$-terms to get $\overline{\lambda^{\to}}$-terms.

Realizability for $\lambda^{\to}$, II

We use *untyped* closed lambda-terms as realizers. This works because the reductions are *type-oblivious*. We define a realizability relation between realizers and formulas:

$$
\begin{aligned}
M \Vdash p &\text{ iff } M \downarrow \\
M \Vdash \bot &\text{ iff } \bot \\
M \Vdash \varphi \wedge \psi &\text{ iff } M \downarrow \langle M_1, M_2 \rangle \wedge (M_1 \Vdash \varphi) \wedge (M_2 \Vdash \psi) \\
M \Vdash \varphi \vee \psi &\text{ iff } (M \downarrow \mathsf{inl}(M_1) \wedge M_1 \Vdash \varphi) \\
&\qquad \vee (M \downarrow \mathsf{inr}(M_2) \wedge M_2 \Vdash \psi) \\
M \Vdash \varphi \to \psi &\text{ iff } (M \downarrow \lambda x, M_1) \wedge \forall N, (N \Vdash \varphi) \to (M_1[N/x] \Vdash \psi)
\end{aligned}
$$

Normalization for $\lambda^{\to}$

A realizability environment ρ is partial function from proof variables to realizers. Write $\rho \vDash \Gamma$ if $\rho(x) \Vdash \psi$ for $(x{:}\psi) \in \Gamma$.

Theorem

If $\Gamma \vdash M{:}\varphi$, *then for all* $\rho \vDash \Gamma$, *we have* $\overline{M}[\rho] \Vdash \varphi$.

Corollaries:

Normalization: If $\vdash M{:}\varphi$, then M normalizes.

IPC is consistent: There is no M with $\vdash M{:}\bot$.

The disjunction property for IPC: If $\vdash \varphi \vee \psi$, then $\vdash \varphi$ or $\vdash \psi$.

Set theory

Moczydłowski studies Intuitionistic ZF with Replacement. Terms and formulas are defined by a mutual grammar:

$$
\begin{aligned}
t ::= {} & a \mid \emptyset \mid \{t, t\} \mid \omega \mid \bigcup t \mid P(t) \\
& \mid S_{\varphi(a,\vec{f})}(t, \vec{t}) \mid R_{\varphi(a,b,\vec{f})}(t, \vec{t}) \\
\varphi ::= {} & \bot \mid t \in t \mid t = t \mid t \in_I t \\
& \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \varphi \to \varphi \\
& \mid \forall a, \varphi \mid \exists a, \varphi
\end{aligned}
$$

Here:

$$
\begin{aligned}
S_{\varphi(a,\vec{f})}(t, \vec{t}) &\equiv \{ v \in t \mid \varphi(c, \vec{t}) \} \\
R_{\varphi(a,b,\vec{f})}(t, \vec{t}) &\equiv \{ c \mid (\forall x \in t, \exists! y, \varphi(x, y, \vec{t})) \wedge (\exists x \in t, \varphi(x, c, \vec{t})) \}
\end{aligned}
$$

Axioms of IZF_R

(IN)	$a \in b \leftrightarrow \exists c, c \in_I b \wedge c = a$
(EQ)	$a = b \leftrightarrow \forall d, (d \in_I a \to d \in b) \wedge (d \in_I b \to d \in a)$
(EMPTY)	$c \in_I \emptyset \leftrightarrow \bot$
(PAIR)	$c \in_I \{a, b\} \leftrightarrow c = a \vee c = b$
(INF)	$c \in_I \omega \leftrightarrow c = \emptyset \vee \exists b \in \omega, c = S(b)$
($\mathrm{SEP}_{\varphi(a,\vec{f})}$)	$c \in_I S_{\varphi(a,\vec{f})}(a, \vec{f}) \leftrightarrow c \in a \wedge \varphi(c, \vec{f})$
(UNION)	$c \in_I \bigcup a \leftrightarrow \exists b \in a, c \in b$
(POWER)	$c \in_I P(a) \leftrightarrow \forall b \in c, b \in a$

continues ...

Axioms of IZF_R, II

continued:

$$(\mathrm{REPL}_{\varphi(a,b,\vec{f})}) \quad c \in_I R_{\varphi(a,b,\vec{f})}(a,\vec{f}) \leftrightarrow (\forall x \in a, \exists! y, \varphi(x,y,\vec{f})) \wedge (\exists x \in a, \varphi(x,c,\vec{f}))$$

$$(\mathrm{IND}_{\varphi(a,\vec{f})}) \quad (\forall a, (\forall b \in_I a, \varphi(b,\vec{f})) \to \varphi(a,\vec{f})) \to \forall a, \varphi(a,\vec{f})$$

The λZ calculus

We use two sets of variables, *proof*- and *set*-variables.
Terms:

$$
\begin{aligned}
M ::= x \mid M\ t \mid M\ N \mid \lambda a, M \mid \lambda x{:}\varphi, M \\
\mid \mathsf{inl}(M) \mid \mathsf{inr}(M) \mid \mathsf{case}\, M\, \mathsf{of} \begin{pmatrix} \mathsf{inl}\, x{:}\varphi \Rightarrow N_1 \\ \mathsf{inr}\, y{:}\psi \Rightarrow N_2 \end{pmatrix} \\
\mid \langle M, N \rangle \mid \mathsf{fst}(M) \mid \mathsf{snd}(M) \mid \mathsf{magic}(M) \\
\mid [t, M] \mid \mathsf{let}\, [a, x{:}\varphi] := M\, \mathsf{in}\, N
\end{aligned}
$$

continues . . .

The λZ calculus, II

$$
\begin{aligned}
M ::= \quad & \ldots \mid \mathsf{inProp}(t, u, M) \mid \mathsf{inRep}(t, u, M) \\
& \mid \mathsf{eqProp}(t, u, M) \mid \mathsf{eqRep}(t, u, M) \\
& \mid \mathsf{pairProp}(t, u_1, u_2, M) \mid \mathsf{pairRep}(t, u_1, u_2, M) \\
& \mid \mathsf{unionProp}(t, u, M) \mid \mathsf{unionRep}(t, u, M) \\
& \mid \mathsf{sep}_{\varphi(a,\vec{f})}\mathsf{Prop}(t, u, \vec{u}, M) \mid \mathsf{sep}_{\varphi(a,\vec{f})}\mathsf{Rep}(t, u, \vec{u}, M) \\
& \mid \mathsf{powerProp}(t, u, M) \mid \mathsf{powerRep}(t, u, M) \\
& \mid \mathsf{infProp}(t, M) \mid \mathsf{infRep}(t, M) \\
& \mid \mathsf{repl}_{\varphi(a,b,\vec{f})}\mathsf{Prop}(t, u, \vec{u}, M) \mid \mathsf{repl}_{\varphi(a,b,\vec{f})}\mathsf{Rep}(t, u, \vec{u}, M) \\
& \mid \mathsf{ind}_{\varphi(a,\vec{f})}(M, \vec{t})
\end{aligned}
$$

The λZ calculus, III

We'll abbreviate the -Prop and -Rep-axioms as

$$\mathrm{axProp}(t, \vec{u}, M) \mid \mathrm{axRep}(t, \vec{u}, M),$$

where the length of $\vec{u}$ depends on the axiom.

Typing for the λZ calculus

Same rules as for IPC plus first-order rules:

$$\frac{\Gamma \vdash M{:}\varphi}{\Gamma \vdash (\lambda a, M){:}\forall a, \varphi}\ a \notin \mathsf{FV}_s(\Gamma) \qquad \frac{\Gamma \vdash M{:}\forall a, \varphi}{\Gamma \vdash M\ t{:}\varphi[t/a]}$$

$$\frac{\Gamma \vdash M{:}\varphi[t/a]}{\Gamma \vdash [t, M]{:}\exists a, \varphi}$$

$$\frac{\Gamma \vdash M{:}\exists a, \varphi \qquad \Gamma, x{:}\varphi \vdash N{:}\psi}{\Gamma \vdash (\mathsf{let}\ [a, x{:}\varphi] := M\ \mathsf{in}\ N){:}\psi}\ a \notin \mathsf{FV}_s(\Gamma, \psi)$$

continues . . .

Typing for the λZ calculus, II

Plus rules for the axioms, first equality (EQ):

$$\frac{\Gamma \vdash M{:}\forall d, (d \in_I t \to d \in u) \wedge (d \in_I u \to d \in t)}{\Gamma \vdash \mathsf{eqRep}(t, u, M){:}t = u}$$

$$\frac{\Gamma \vdash M{:}t = u}{\Gamma \vdash \mathsf{eqProp}(t, u, M){:}\forall d, (d \in_I t \to d \in u) \wedge (d \in_I u \to d \in t)}$$

continues . . .

Typing for the λZ calculus, III

Then, membership (IN):

$$\frac{\Gamma \vdash M{:}\exists c, c \in_I u \wedge t = c}{\Gamma \vdash \mathsf{inRep}(t, u, M){:}t \in u}$$

$$\frac{\Gamma \vdash M{:}t \in u}{\Gamma \vdash \mathsf{inProp}(t, u, M){:}\exists c, c \in_I u \wedge t = c}$$

continues . . .

Typing for the λZ calculus, IV

The other axioms all follow the same pattern:

$$\frac{\Gamma \vdash M{:}\varphi_A(t, \vec{u})}{\Gamma \vdash \mathsf{axRep}(t, \vec{u}, M){:}t \in_I t_A(\vec{u})}$$

$$\frac{\Gamma \vdash M{:}t \in_I t_A(\vec{u})}{\Gamma \vdash \mathsf{axProp}(t, \vec{u}, M){:}\varphi_A(t, \vec{u})}$$

continues . . .

Typing for the λZ calculus, V

Except $\in_I$-induction (IND):

$$\frac{\Gamma \vdash M{:}\forall c, (\forall b, b \in_I c \to \varphi(b, \vec{t})) \to \varphi(c, \vec{t})}{\Gamma \vdash \mathsf{ind}_{\varphi(a,\vec{f})}(M, \vec{t}){:}\forall a, \varphi(a, \vec{t})}$$

Reduction for the λZ calculus

Same reductions as for IPC plus:

$$
\begin{aligned}
(\lambda a, M)\ t &\longrightarrow M[t/a] \\
\mathsf{let}\ [a, x{:}\varphi] := [t, M]\ \mathsf{in}\ N &\longrightarrow N[t/a][M/x] \\
\mathsf{axProp}(t, \vec{u}, \mathsf{axRep}(t, \vec{u}, M)) &\longrightarrow M \\
\mathsf{ind}_{\varphi(a,\vec{f})}(M, \vec{t}) &\longrightarrow \lambda c, M\ c\ (\lambda b, \lambda x{:}(b \in_I c), \\
&\qquad\qquad \mathsf{ind}_{\varphi(a,\vec{f})}(M, \vec{t})b)
\end{aligned}
$$

Values:

$$
\begin{aligned}
V ::= {} & \lambda a, M \mid \lambda x{:}\varphi, M \mid \mathsf{inr}(M) \mid \mathsf{inr}(M) \\
& \mid [t, M] \mid \langle M, N \rangle \mid \mathsf{axRep}(t, \vec{u}, M)
\end{aligned}
$$

Realizability for the λZ calculus

Realizability for IZF was first defined by David McCarty in his 1984 PhD-thesis. Moczydłowski builds on this work to prove normalization.
To get realizers, we erase types and sets from λZ-terms to get $\lambda\overline{Z}$-terms (all sets disappear or become $\emptyset$):

$$\begin{aligned} M ::= {} & x \mid M\,\emptyset \mid M\,N \mid \lambda a, M \mid \lambda x, M \\ & \mid \mathsf{inl}(M) \mid \mathsf{inr}(M) \mid \mathsf{case}\, M \,\mathsf{of} \begin{pmatrix} \mathsf{inl}\, x \Rightarrow N_1 \\ \mathsf{inr}\, y \Rightarrow N_2 \end{pmatrix} \\ & \mid \langle M, N \rangle \mid \mathsf{fst}(M) \mid \mathsf{snd}(M) \mid \mathsf{magic}(M) \\ & \mid [\emptyset, M] \mid \mathsf{let}\, [a, x] := M \,\mathsf{in}\, N \\ & \mid \mathsf{axProp}(M) \mid \mathsf{axRep}(M) \mid \mathsf{ind}(M) \end{aligned}$$

Lambda names

The idea is to have a version of the cumulative hierarchy that for each set includes realizers as evidence for the placement in the hiearchy.

Definition

A λ-name is a set of pairs (v, B) where $v \in \lambda\overline{Z}_{vc}$ and B is a λ-name. The class of λ-names is denoted V^λ.
We have

$$V^\lambda = \bigcup_{\alpha \in \mathrm{Ord}} V^\lambda_\alpha, \qquad V^\lambda_\alpha = \bigcup_{\beta < \alpha} \mathrm{P}(\lambda\overline{Z}_{vc} \times V^\lambda_\beta),$$

and for a λ-name A we let $\lambda\mathrm{rk}(A)$ denote the smallest ordinal α with $A \in V^\lambda_\alpha$.

Prerealizability

Define $M \Vdash A \in_I B$, $M \Vdash A \in B$ and $M \Vdash A = B$ for $M \in \lambda\overline{Z}_c$, and $A, B \in V^\lambda$:

$$
\begin{aligned}
M \Vdash A \in_I B &\equiv M \downarrow v \wedge (v, A) \in B \\
M \Vdash A \in B &\equiv M \downarrow \mathsf{inRep}(N) \wedge N \downarrow [\emptyset, O] \\
&\quad \wedge \exists C \in V^\lambda, O \downarrow \langle O_1, O_2 \rangle \\
&\quad \wedge O_1 \Vdash C \in_I B \wedge O_2 \Vdash A = C \\
M \Vdash A = B &\equiv M \downarrow \mathsf{eqRep}(M_0) \wedge M_0 \downarrow \lambda a, M_1 \\
&\quad \forall D \in V^\lambda, M_1[\emptyset/a] \downarrow \langle O, P \rangle \\
&\qquad \wedge O \downarrow \lambda x, O_1 \wedge (\forall N \Vdash D \in_I A, O_1[N/x] \Vdash D \in B) \\
&\qquad \wedge P \downarrow \lambda x, P_1 \wedge (\forall N \Vdash D \in_I A, P_1[N/x] \Vdash D \in A)
\end{aligned}
$$

Enriched language

Definition

For $C \in V^\lambda$, let $C^+ \equiv \{(M, A) \mid M \Vdash A \in C\}$.

Definition

Let $\mathrm{L}(V^\lambda)$ be the first-order language obtained by enriching the signature of $\mathrm{IZF_R}$ with constants for all λ-names.

Definition

A realizabilty environment ρ is a partial function from variables in $\mathrm{L}(V^\lambda)$ to the class of λ-names, V^λ.

Realizability for IZF_R

We define by mutual induction, for φ a formula of $L(V^\lambda)$, a term t of $L(V^\lambda)$, and for an environment ρ defined on the free variables in φ or t, a realizability relation $M \Vdash_\rho \varphi$ (for $M \in \lambda\overline{Z}_c$), and a denotation $[\![t]\!]_\rho \in V^\lambda$.

$$[\![a]\!]_\rho \equiv \rho(a)$$

$$[\![A]\!]_\rho \equiv A$$

$[\![\omega]\!]_\rho \equiv \omega'$ (a suitable λ-name for ω)

$[\![t_A(\vec{u})]\!]_\rho \equiv \{(\mathrm{axRep}(N), B) \in \lambda\overline{Z}_{vc} \times V^\lambda_\gamma \mid N \Vdash_\rho \varphi_A(B, [\![\vec{u}]\!]_\rho)\}$
(for a suitable ordinal γ depending on the λ-ranks of the parameters to the axiom)

Realizability for IZF_R, II

$$
\begin{array}{ll}
M \Vdash_\rho \bot & \text{iff } \bot \\
M \Vdash_\rho t \in_I s & \text{iff } M \Vdash [\![t]\!]_\rho \in_I [\![s]\!]_\rho \\
M \Vdash_\rho t \in s & \text{iff } M \Vdash [\![t]\!]_\rho \in [\![s]\!]_\rho \\
M \Vdash_\rho t = s & \text{iff } M \Vdash [\![t]\!]_\rho = [\![s]\!]_\rho \\
M \Vdash_\rho \varphi \wedge \psi & \text{iff } M \downarrow \langle M_1, M_2 \rangle \wedge (M_1 \Vdash_\rho \varphi) \wedge (M_2 \Vdash_\rho \psi) \\
M \Vdash_\rho \varphi \vee \psi & \text{iff } (M \downarrow \mathsf{inl}(M_1) \wedge M_1 \Vdash_\rho \varphi) \\
& \qquad \vee (M \downarrow \mathsf{inr}(M_2) \wedge M_2 \Vdash_\rho \psi) \\
M \Vdash_\rho \varphi \to \psi & \text{iff } (M \downarrow \lambda x, M_1) \wedge \forall N \Vdash_\rho \varphi, M_1[N/x] \Vdash_\rho \psi \\
M \Vdash_\rho \exists a, \varphi & \text{iff } M \downarrow [\emptyset, N] \wedge \exists A \in V^\lambda, N \Vdash_\rho \varphi[A/a] \\
M \Vdash_\rho \forall a, \varphi & \text{iff } M \downarrow \lambda a, N \wedge \forall A \in V^\lambda, N[\emptyset/a] \Vdash_\rho \varphi[A/a]
\end{array}
$$

Properties of this interpretation

Lemma

If $A \in V^{\lambda}_{\alpha}$, *then there is a* $\beta < \alpha$, *so that if* $M \Vdash B \in A$, *then* $B \in V^{\lambda}_{\beta}$.
If $M \Vdash B = A$, *then* $B \in V^{\lambda}_{\alpha}$.
If $M \Vdash B \in_I A$, *then* $\lambda rk B < \lambda rk A$.

Lemma

For any intensional axiom we have

$$(M, C) \in [\![t_A(\vec{u})]\!]_\rho \quad \textit{iff} \quad M = \mathsf{axRep}(N) \textit{ and } N \Vdash_\rho \varphi_A(C, [\![\vec{u}]\!]_\rho)$$

Normalization of λZ

We write $\rho \models \Gamma \vdash M{:}\varphi$ if ρ assigns lambda-names to free first-order variables and realizers to context proof variables, so that for $(x{:}\psi) \in \Gamma$, we have $\rho(x) \Vdash_\rho \psi$.

Theorem

If $\Gamma \vdash M{:}\varphi$*, then for all* $\rho \models \Gamma \vdash M{:}\varphi$*, we have* $\overline{M}[\rho] \Vdash_\rho \varphi$*.*

Corollary

If $\vdash M{:}\varphi$*, then* M *normalizes.*

Applications

Corollary

If $IZF_R \vdash \varphi \vee \psi$*, then* $IZF_R \vdash \varphi$ *or* $IZF_R \vdash \psi$*.*

Corollary

If $IZF_R \vdash \exists x, \varphi(x)$*, then there is a term* t *so that* $IZF_R \vdash \varphi(t)$*.*

Corollary

If $IZF_R \vdash \exists x \in \omega, \varphi(x)$*, then there is number* n *so that* $IZF_R \vdash \varphi(\bar{n})$*.*

Failure of strong normalization

An obstacle to strong normalization of intuitionistic set theories is Crabbé's Counterexample:

Let $t = \{ x \in \emptyset \mid x \in x \to \bot \}$. There is a term $M{:}(t \in \emptyset \to \bot)$ that does not normalize if we allow reductions under the binder.

Extensions

Moczydłowski's approach extends to give normalizing calculi for IZF_R with countably many inaccessibles. He also gives a dependent set theory that proves collection (which is stronger than replacement, intuitionistically).

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