UNORDERED PAIRS IN HOMOTOPY TYPE THEORY

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Abstract. We give two different definitions of unordered pairs in a type in the context of homotopy type theory and univalent foundations. One is the homotopy orbit type of the Σ_2-action on the of ordered pairs, the other the is categorical orbit type. We prove that the categorical version applied to a set is still a set, answering a question of Capriotti. We also explain how the situation with triples is more troublesome, and why general coherent unordered multisets are out of reach at the moment.

1. Introduction

Homotopy type theory (HoTT) is Martin-Löf dependent type theory extended with Voevodsky's univalence axiom and higher inductive types [11]. Its types are interpreted as ∞-groupoids, perhaps structured or varying, in the sense of lying in a Grothendieck $(\infty, 1)$-topos. More precisely, HoTT can be interpreted in simplicial sets [6], which have a model structure presenting the $(\infty, 1)$-topos of ordinary ∞-groupoids, as well as in model structures presenting an arbitrary $(\infty, 1)$-topos [9].

HoTT can serve as a foundational system, and among other constructions, it features ordered pairs (as elements of product types $A \times B$ or even dependent sum types $(x : A) \times B(x)$).[1] In this paper, we investigate some ways of constructing types of *unordered* pairs $\{x_0, x_1\}$, where x_0, x_1 range over a fixed type X. Crucially, we do not insist that X is a set, i.e., a type whose identity types $x_0 =_X x_1$ are propositions, where a proposition is a type P, all of whose elements can be identified via a construction of type $(p_0, p_1 : P) \to p_0 =_P p_1$.[2]

In Section 2, we discuss the notion of homotopy unordered pairs in X, $\mathrm{hUP}(X)$, which consist of two elements of X picked out by an arbitrary two-element type. There we also show that the set-truncation of $\mathrm{hUP}(X)$ is the set of two-element *sub*sets of X.

However, even if X is a set, $\mathrm{hUP}(X)$ may not be a set. This phenomenon is familiar from equivariant homotopy theory, where there are two versions of Σ_2-equivariant homotopy types. One is the pure homotopy version, represented by the $(\infty, 1)$-topos of presheaves on $\mathrm{B}\Sigma_2$ where $\mathrm{B}\Sigma_2$ is the classifying space of Σ_2. This is captured in HoTT as type families over $\mathrm{B}\Sigma_2$, $A : \mathrm{B}\Sigma_2 \to \mathcal{U}$. The other is the genuine equivariant version, which thanks to Elmendorf's theorem [4] is represented by the $(\infty, 1)$-presheaf topos over the *orbit category* of Σ_2. In Section 3 we show how to capture this in HoTT, following [10], and we introduce the resulting notion of genuine unordered pairs in X, $\mathrm{UP}(X)$. The definition relies on a homotopy colimit

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[1] Here we shall write Σ-types as shown, and Π-types similarly as $(x : A) \to B(x)$, instead of $\sum_{x:A} B(x)$ and $\prod_{x:A} B(x)$, respectively.

[2] From now on we'll assume familiarity with the foundations of HoTT as laid out in [11].

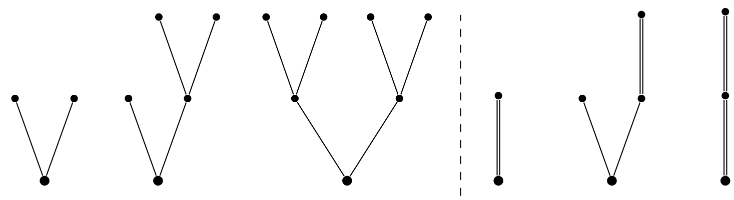

FIGURE 1. Two kinds of unordered rooted binary trees

calculation, theorem 3.1, that I believe is a new, though fairly easy, observation, even in the setting of classical homotopy theory.

Here we also answer a question of Capriotti by showing that if X is a set, then $\mathrm{UP}(X)$ is a set. Hence for sets, the genuine unordered pairs, the set quotient of the homotopy unordered pairs, and the type of two-element subsets of X all agree.

The difference between the homotopy and the genuine unordered pairs can be visually explained in terms of the corresponding types of (unlabeled) unordered rooted binary trees, hBTree and BTree. Both are inductive types with a nullary leaf constructor leaf and constructor node with either of the following types:

$$\mathrm{hUP}(\mathrm{hBTree}) \to \mathrm{hBTree}, \qquad \mathrm{UP}(\mathrm{BTree}) \to \mathrm{BTree}\,.$$

In hBTree, the tree with two identical leaves, $\mathrm{node}(\{\mathrm{leaf}, \mathrm{leaf}\})$, has an automorphism of order two, while in BTree, it has no automorphisms. This is illustrated in Fig. 1, where the three trees as displayed left of the dashed line as elements of hBTree and right of the dashed line as elements of BTree. The difference is, that when a node has two identifiable subtrees as children, then in BTree we don't get any automorphisms, so we draw the children as one subtree connected with a double line. Indeed, it will follow that BTree is a set, while hBTree isn't. Which version is desired depends on the intended application.

1.1. **Precise setting.** In this paper we shall work in a rather basic HoTT-setting, based on predicative Martin-Löf type theory with a hierarchy of universes $\mathcal{U}_0 : \mathcal{U}_1 : \cdots$, each satisfying the univalence axiom and each closed under the usual type formers as well as pushouts. We shall mostly only use universe, though, so we simply write $\mathcal{U}$ without an index.

From pushouts we can define many other finitary higher inductive types, including all n-truncations [8].

1.2. **Related work.** For a recent survey of equivariant homotopy theory, see [5]. The type of two-element types was used in [2] to define the homotopy types of the real projective spaces $\mathbb{R}\mathrm{P}^n$ and in [3] to study a fragment of a programming language for reversible computing. Unordered trees and finite multisets were studied in terms of set-quotient inductive types in [7].

2. Homotopy unordered pairs

We recall from [2] some facts about the type of two-element types. We define $\mathrm{B}\Sigma_2 :\equiv (A : \mathcal{U}) \times \|A \simeq \mathbf{2}\|$, where $\mathbf{2}$ is the canonical two-element type, with elements denoted 0 and 1. The type $\mathrm{B}\Sigma_2$ is the connected component of the universe $\mathcal{U}$ is $\mathbf{2}$, it is pointed at $\mathbf{2}$, and its loop space there is equivalent, as a group, to the symmetric group on two elements, Σ_2. It is thus a classifying type (in the sense of [1]) for Σ_2, which accounts for our notation. We write $\mathrm{El} : \mathrm{B}\Sigma_2 \to \mathcal{U}$ for the first projection.

Definition 2.1. For any type X, the type of *homotopy unordered pairs* in X is

$$\mathrm{hUP}(X) :\equiv (K : \mathrm{B}\Sigma_2) \times (\mathrm{El}\, K \to X).$$

That is, the type $\mathrm{hUP}(X)$ consists of a choice of a two element type K and function from the elements of K to X.

Sometimes, this is exactly what we want. For instance, that a binary operation $\mu : X \times X \to X$ is coherently commutative can be witnessed by an extension of μ to $\mathrm{hUP}(X)$, relative to the map $p : X \times X \to \mathrm{hUP}(X)$ that sends (x_0, x_1) to the homotopy unordered pair labeled by $\mathbf{2}$, sending 0 to x_0 and 1 to x_1.

From the definition, it is clear that $\mathrm{hUP}(\mathbf{1}) \simeq \mathrm{B}\Sigma_2$, where $\mathbf{1}$ denotes the unit type. In particular, $\mathrm{hUP}(\mathbf{1})$ is not a set, even though $\mathbf{1}$ is.

One way to understand definition 2.1 is via the Σ_2-action on the type of ordered pairs in X, $X \times X$. This is defined via the type family

$$\mathrm{hEP}(X) :\equiv K \mapsto (\mathrm{El}\, K \to X) : \mathrm{B}\Sigma_2 \to \mathcal{U}, \tag{1}$$

which indeed represents an action on $X \times X$ via the canonical equivalence $(\mathbf{2} \to X) \simeq X \times X$. The non-trivial element of Σ_2 maps a pair (x_0, x_1) to (x_1, x_0).[3] For any higher group G, we have an adjoint triple

$$(BG \to \mathcal{U}) \;\underset{\underset{(-)^{hG}}{\longrightarrow}}{\overset{\overset{(-)_{hG}}{\longrightarrow}}{\xleftarrow{\;\perp\;\Delta\;\perp\;}}}\; \mathcal{U}$$

relating the homotopy orbit and fixed point types to the trivial actions map. By definition of the homotopy orbit type [1, Sec. 4.2] as the Σ-type of $\mathrm{hEP}(X)$, we have $\mathrm{hEP}(X)_{\mathrm{h}\Sigma_2} \equiv \mathrm{hUP}(X)$. As for the homotopy fixed point type, $\mathrm{hEP}(X)^{\mathrm{h}\Sigma_2} \equiv (K : \mathrm{B}\Sigma_2) \to (\mathrm{El}\, K \to X)$, we have the following.

Lemma 2.2. *The map $s : X \to \mathrm{hEP}(X)^{\mathrm{h}\Sigma_2}$ that sends $x : X$ to the dependent function that sends $K : \mathrm{B}\Sigma_2$ to the constant function at x is an equivalence.*

Proof. Recall from [2, Thm. II.2] that the type of pointed two-element types is contractible. This means we can factor s as the following sequence of equivalences:

$$\begin{aligned} X &\simeq \big((K : \mathrm{B}\Sigma_2) \times \mathrm{El}\, K\big) \to X \\ &\simeq (K : \mathrm{B}\Sigma_2) \to (\mathrm{El}\, K \to X) \equiv \mathrm{hEP}(X)^{\mathrm{h}\Sigma_2}. \end{aligned}$$ □

So far, we have ignored universe levels, but it is clear that if X is a small type, i.e., a type in $\mathcal{U}$, then $\mathrm{hUP}(X)$ belong to a larger universe. However, this is only a cosmetic defect.

Lemma 2.3. *The type $\mathrm{hUP}(X)$ is essentially $\mathcal{U}$-small for any $X : \mathcal{U}$.*

Recall that a type is essentially $\mathcal{U}$-small if it is equivalent to a type in $\mathcal{U}$. By the univalence axiom for $\mathcal{U}$, this is a proposition.

Proof. There are two ways to see this, each relying on the type-theoretic replacement principle, which is a corollary of Rijke's join construction [8]. This says that if $f : A \to B$ is a map from a $\mathcal{U}$-small type A to an locally $\mathcal{U}$-small type B (i.e.,

[3] To ease readability, we write (x_0, x_1) also for the map $\mathbf{2} \to X$ determined by x_0 and x_1.

one whose identity types are essentially $\mathcal{U}$-small), then the image of f is essentially $\mathcal{U}$-small. In particular, if f additionally surjective, then B itself is essentially $\mathcal{U}$-small.

So the first way is to observe that $\mathrm{B}\Sigma_2$ is a pointed connected locally $\mathcal{U}$-small type and hence essentially $\mathcal{U}$-small, so $\mathrm{hUP}(X)$ is as well.

The second way is to observe that $\mathrm{hUP}(X)$ is locally $\mathcal{U}$-small, and $p : X \times X \to \mathrm{hUP}(X)$ is a surjection from a $\mathcal{U}$-small type, hence $\mathrm{hUP}(X)$ is essentially $\mathcal{U}$-small. □

To finish this section, we shall briefly look at the set truncation of $\mathrm{hUP}(X)$. This turns out to coincide with another common set-theoretical construction of the unordered pairs in a type X, namely as the set $\mathrm{sUP}(X)$ of those subsets of X that have the form $\{x_0, x_1\}$ for some $x_0, x_1 : X$. Here, the subset $\{x_0, x_1\}$ is conceived as the predicate $P_{x_0,x_1} : X \to \mathrm{Prop}$ defined by $P_{x_0,x_1}(x) :\equiv \|x = x_0 + x = x_1\|$. Then we set

$$\mathrm{sUP}(X) :\equiv (P : X \to \mathrm{Prop}) \times \exists_{x_0,x_1:X}(P = \{x_0, x_1\}).$$

This is a set, since Prop is a set, and it's essentially $\mathcal{U}$-small since it's locally $\mathcal{U}$-small and the map $p : X \times X \to \mathrm{sUP}(X)$ sending (x_0, x_1) is $\{x_0, x_1\}$ is clearly surjective.

Next we see that p extends to $\mathrm{hUP}(X)$: To define $q : \mathrm{hUP}(X) \to \mathrm{sUP}(X)$, it suffices to define a function

$$\tilde{q} : (A : \mathcal{U}) \to (A \to X) \to (\mathbf{2} \simeq A) \to \mathrm{hUP}(X)$$

such that for each $A : \mathcal{U}$ and $f : A \to X$, the function $\tilde{q}(A, f) : (\mathbf{2} \simeq A) \to \mathrm{hUP}(X)$ is weakly constant, because then $\tilde{q}(A, f)$ factors through $\|\mathbf{2} \simeq A\|$. We define $\tilde{q}$ in the obvious way, sending (A, f, e) to $p(f \circ e)$. This is weakly constant in e, because if have some other $e : \mathbf{2} \simeq A$, then $e' \circ e^{-1} : \mathbf{2} \simeq \mathbf{2}$ is either the identity or the swap, and $\{x_0, x_1\} = \{x_1, x_0\}$ in $\mathrm{sUP}(X)$ for any $x_0, x_1 : X$.

Lemma 2.4. *The map* $q : \mathrm{hUP}(X) \to \mathrm{sUP}(X)$ *exhibits* $\mathrm{sUP}(X)$ *as the set truncation of* $\mathrm{hUP}(X)$.

Proof. Note that q is surjective as an extension of the surjection $p : X \times X \to \mathrm{sUP}(X)$. Since being connected is a proposition, to show that the fibers are connected, it suffices to look at fibers over elements $\{x_0, x_1\} : \mathrm{sUP}(X)$ for $x_0, x_1 : X$. Let (K, f) and (K', f') be two elements of $\mathrm{hUP}(X)$ lying in the fiber over $\{x_0, x_1\}$. To show that they are merely equal, we may assume that the domains of f and f' are identified with $\mathbf{2}$. Then the images under q can be described as $\{y_0, y_1\}$ and $\{y_0', y_1'\}$, respectively, for some $y_0, y_0', y_1, y_1' : X$, such that

$$\{x_0, x_1\} = \{y_0, y_1\} = \{y_0', y_1'\}$$

in $\mathrm{sUP}(X)$. This means that for every $x : X$ we have a logical equivalence

$$\|x = y_0 + x = y_1\| \leftrightarrow \|x = y_0' + x = y_1'\|.$$

Still using the fact that we're proving a proposition, we apply the left-to-right direction for $x :\equiv y_0$ and $x :\equiv y_1$ and the right-to-left direction for $x :\equiv y_0'$ and $x :\equiv y_1'$ to obtain sixteen cases, which divide into three different forms:

$$(1)\ y_0 = y_1 = y_0' = y_1'; \quad (2)\ y_0 = y_0', y_1 = y_1'; \quad (3)\ y_0 = y_1', y_1 = y_0'.$$

In each case, we find an identification $\mathbf{2} \simeq \mathbf{2}$ relative to which (K, f) and (K', f') can be identified.

We have shown that q is a 0-connected map to a set, hence it induces an equivalence $\|\mathrm{hUP}(X)\|_0 \simeq \mathrm{sUP}(X)$, as desired. □

3. Genuine unordered pairs

3.1. **Equivariant homotopy theory.** To give our second definition of unordered pairs, the genuine unordered pairs, we need to recall a few facts about genuine equivariant homotopy theory. To this purpose, fix a finite group G. The classical genuine G-equivariant homotopy theory arises from the category of (nice) topological spaces with a G-action and G-equivariant maps. Any G-equivariant map $f : X \to Y$ restricts to a map of H-fixed points, $f^H : X^H \to Y^H$, for any subgroup H of G, where $X^H := \{\, x \in X \mid hx = x \text{ for all } h \in H \,\}$. This category has a model structure where the weak equivalences are those maps f for which each f^H is a weak equivalence, as H ranges over the subgroup of G.

A priori, the resulting $(\infty, 1)$-category $\mathcal{S}_G$ does not seem easy to describe in terms of the $(\infty, 1)$-category of ∞-groupoids (one model of which is the usual Kan–Quillen model category structure on nice topological spaces). It is not even clear that it is an $(\infty, 1)$-topos. However, if we "probe" the G-spaces X by a small collection O_G of certain nice G-spaces, we can hope to relate $\mathcal{S}_G$ to the $(\infty, 1)$-topos of presheaves on the full subcategory spanned by O_G.

Note that the maps $G/H \to X$ are exactly X^H, so it's a good idea to try to probe with the orbit spaces G/H. In fact, it turns out that if we take O_G to consist of the G-orbit spaces G/H, for H a subgroup of G, we get an equivalence of $(\infty, 1)$-categories $\mathcal{S}_G \simeq \mathrm{PSh}(\mathrm{Orb}_G)$, where Orb_G is the *orbit category* of G. This is the content of Elmendorf's theorem [4].

So the orbit category of G, Orb_G, is presented by a strict 1-category whose objects are G/H and whose morphisms $G/H_1 \to G/H_2$ are the G-equivariant maps. We can give a more direct description of the groupoid of objects of Orb_G in HoTT, using the fact that Orb_G is a full subcategory of the category of G-sets, whose objects are type families of sets over BG, $X : BG \to \mathrm{Set}$, and whose arrows $X \to Y$ are transformations over BG, $f : (t : BG) \to X(t) \to Y(t)$. Then Orb_G is spanned by those X whose total space (i.e., orbit groupoid) is connected. Classically, each such G-set is has decidable equality, so each subgroup has finite index. This doesn't necessarily hold in a constructive setting, however, it seems reasonable even from a constructive point-of-view to restrict to the finite index subgroups.[4] This is equivalent, for a finite group G, to requiring the underlying sets of the spaces $X(\mathrm{pt})$ to have decidable equality.

Note that $\mathrm{Hom}_{\mathrm{Orb}_G}(G/H, G/K) = (G/K)^H = \{\, gK \in G/K \mid g^{-1}Hg \subseteq K \,\}$ consists of those cosets that conjugate H into K. In particular, the hom-set is empty unless H is conjugate to a subgroup of K, and any endomorphism of G/H is invertible, the automorphism group being isomorphic to $N_G(H)/H$, i.e., the Weyl group of G relative to H. So we see that Orb_G is a (generalized) direct EI-category, as also noted in [10, Sec. 8.6], with G/G as a terminal object.

[4] I'll leave it to some other occasion to investigate how, constructively, *super-genuine* equivariant homotopy theory, taking into account *all* subgroups of G, differs from the merely *genuine* version discussed here.

3.2. **Orbit category of Σ_2.** Let us examine more closely the orbit category of Σ_2 as seen in HoTT, recalling [10, Example 8.5]. The groupoid of objects of Orb_{Σ_2} is

$$\begin{aligned}\mathrm{Orb}^{\mathrm{core}}_{\Sigma_2} &:= (X : \mathrm{B}\Sigma_2 \to \mathrm{Set}) \times \mathrm{isConn}_0((J : \mathrm{B}\Sigma_2) \times X\,J) \times \mathrm{hasDecEq}(X\,\mathrm{pt}) \\ &= (X : \mathrm{B}\Sigma_2 \to \mathcal{U}) \times \Big(\|X = \mathrm{El}\| + \|X = \mathrm{Triv}\|\Big) \\ &= \mathrm{B}\Sigma_2 + 1,\end{aligned}$$

where

$$\begin{aligned}\mathrm{El}(J) &= J = (\text{the type of elements of the 2-element set } J) \\ \mathrm{Triv}(J) &= \mathbf{1} = (\text{the unit type}).\end{aligned}$$

Note that this agrees with our earlier description in terms of Weyl groups.

The category structure is completely specified by having the transitive Σ_2-sets in the component of El at level zero with a terminal object at level one, corresponding to the unique transitive Σ_2-set in the component of Triv.

3.3. **Presheaves on direct categories.** Let us now recall from [10] how to define the type of presheaves of types on Orb_{Σ_2} in HoTT, or equivalently, the type of diagrams over $\mathrm{Orb}^{\mathrm{op}}_{\Sigma_2}$. We'll do this in slightly greater generality, by considering the direct category $I^{\triangleright}$ obtained by adding a terminal object (at level one) to a type I (at level zero). The type of presheaves on $I^{\triangleright}$ is the Σ-type $\mathrm{PSh}(I^{\triangleright})$ of pairs A with the following components:

$$\begin{aligned}&A_0 : I \to \mathcal{U} \\ &A_1 : ((i : I) \to A_0\,i) \to \mathcal{U}\end{aligned}$$

The value of A at an object $i : I$ is $A_0\,i$, and A_1 indexes the value at the terminal object over its restrictions to $i : I$. That is, the global sections of A are given by:

$$\Gamma A := (v : (i : I) \to A_0\,i) \times A_1\,v.$$

Let us pause to note that as an $(\infty,1)$-topos, $\mathrm{PSh}(I^{\triangleright})$ is ∞-local, i.e., the adjoint functors in the geometric morphism to $\mathrm{PSh}(1)$ extend to a quadruple of adjoint functors

$$\mathrm{PSh}(I^{\triangleleft}) \;\begin{array}{c} \xrightarrow{\;\Pi\;} \\ \perp \\ \xleftarrow{\;\Delta\;} \\ \perp \\ \xrightarrow{\;\Gamma\;} \\ \perp \\ \xleftarrow[\;\nabla\;]{} \end{array}\; \mathcal{U}$$

where Δ is fully faithful. We've already described the action on objects of Γ concretely. Let us now do the same for Δ, ∇, and Π.

The constant (discrete) presheaf ΔX at a type X has

$$\begin{aligned}\Delta X_0\,i &= X \\ \Delta X_1\,v &= (x : X) \times (v = \mathrm{cst}_x),\end{aligned}$$

where the codiscrete presheaf ∇X has

$$\begin{aligned}\nabla X_0\,i &= 1 \\ \nabla X_1\,v &= X.\end{aligned}$$

To check these assertions, we of course need the type of morphisms of presheaves from A to B. Following [10, Example 8.1], this is the Σ-type with components:

$$\begin{aligned} &f_0 : (i : I) \to A_0\, i \to B_0\, i \\ &f_1 : (v : (i : I) \to A_0\, i) \to A_1\, v \to B_1\, (f_0 \mathbin{\dot\circ} v) \end{aligned}$$

Where $(f_0 \mathbin{\dot\circ} v)(i) :\equiv f_0\, i\, (v\, i)$. For example, we can calculate:

$$\begin{aligned} \mathrm{Hom}_{\mathrm{PSh}(I^{\triangleright})}(\Delta X, A) &\simeq (f_0 : (i : I) \to X \to A_0\, i) \\ &\quad \times \big((v : I \to X) \to (x : X) \to (v = \mathrm{cst}_x) \to A_1(f_0 \mathbin{\dot\circ} v)\big) \\ &\simeq (f_0 : X \to (i : I) \to A_0\, i) \times \big((x : X) \to A_1\, (f_0\, x)\big) \\ &\simeq (X \to \Gamma A) \end{aligned}$$

We elide the definitions of the naturality of these equivalences. Instead, we move on to Π, which as the left adjoint of the diagonal Δ, calculates the colimit of a presheaf.

Theorem 3.1. *The colimit of A, ΠA, can be expressed as the following pushout*

$$\begin{array}{ccc} I \times \Gamma A & \xrightarrow{\mathrm{pr}_2} & \Gamma A \\ \downarrow & & \downarrow \\ (i : I) \times A_0\, i & \longrightarrow & \Pi A \end{array}$$

where the left map takes $(i, (v, w))$ to $(i, v\, i)$.

Proof. Let us abbreviate $A^0 :\equiv (i : I) \to A_0\, i$ so we also have $A_1 : A^0 \to \mathcal{U}$. By the universal property of the colimit, we get:

$$\begin{aligned} \mathrm{Hom}_{\mathrm{PSh}(I^{\triangleright})}(A, \Delta X) &\simeq (f_0 : (i : I) \to A_0\, i \to X) \\ &\quad \times \big((v : A^0) \to A_1\, v \to (x : X) \times (f_0 \mathbin{\dot\circ} v = \mathrm{cst}_x)\big) \\ &\simeq (f_0 : (i : I) \to A_0\, i \to X) \times (g : (v : A^0) \to A_1\, i \to X) \\ &\quad \times \big((v : A^0) \to (w : A_1\, v) \\ &\qquad \to f_0 \mathbin{\dot\circ} v = \mathrm{cst}_{g\, v\, w} \\ &\simeq (f_0 : (i : I) \to A_0\, i \to X) \times (g : (v : A^0) \to A_1\, i \to X) \\ &\quad \times \big((i : I) \to (v : A^0) \to (w : A_1\, v) \\ &\qquad \to (f_0\, i\, (v\, i)) = g\, v\, w \\ &\simeq (\Pi A \to X) \end{aligned}$$

Again, we skip the verification of naturality. □

Notice that the map $\Gamma A \to \Pi A$ is the *points-to-pieces* transformation.

Let us now specialize to the case $I :\equiv BC_p$, where p is a prime. Since the only (detachable) subgroups of C_p are e and C_p itself, we recover the orbit category $\mathrm{Orb}_{C_p} \simeq BC_p^{\triangleright}$. A presheaf A on Orb_{C_p} thus consists of

$$\begin{aligned} &A_0 : BC_p \to \mathcal{U} \\ &A_1 : A^{hC_p} \to \mathcal{U} \end{aligned}$$

where we now identify A^0 with the homotopy fixed point space of the underlying homotopy action $A_0 : BC_p \to \mathcal{U}$. Thus, A_1 describes the type of evidence that a homotopy fixed point is a "genuine" fixed point. (This point was also made in [10, Example 8.6].) The global sections thus give the genuine (also known as "categorical") fixed point type

$$A^{C_p} :\equiv \Gamma A \equiv (v : A^{hC_p}) \times A_1\, v.$$

And as a corollary to theorem 3.1, we identify the genuine orbit type as the pushout:

$$\begin{array}{ccc} BC_p \times A^{C_p} & \xrightarrow{\mathrm{pr}_2} & A^{C_p} \\ \big\downarrow & \ulcorner & \big\downarrow \\ A_{hC_p} & \longrightarrow & A_{C_p} \end{array}$$

Here, we identified the Σ-type $(t : BC_p) \times A_0\, t$ with the homotopy orbit type A_{hC_p}. Thus we see the genuine orbit type as glued together, in this case, from the homotopy orbit type and the genuine fixed point type.

3.4. **Genuine unordered pairs.** We now finally come to our other notion of unordered pairs in HoTT. For this, consider again the homotopy action of Σ_2 (i.e., C_2) on $X \times X$ from (1). We want the genuine fixed points to be X, but in this case, every homotopy fixed point of X^2 is genuine, since

$$(K : \mathrm{B}\Sigma_2) \to (\mathrm{El}\, K \to X) \;\simeq\; \big((K : \mathrm{B}\Sigma_2) \times \mathrm{El}\, K\big) \to X \;\simeq\; X.$$

This means that $\mathrm{EP}(X)$, the type of genuine equivariant pairs in X has:

$$\begin{aligned} \mathrm{EP}(X)_0\, K &= (\mathrm{El}\, K \to X) \\ \mathrm{EP}(X)_1\, v &= 1 \end{aligned}$$

The homotopy orbit type is of course our old notion of homotopy unordered pairs, $\mathrm{EP}(X)_{\mathrm{h}\Sigma_2} = \mathrm{hUP}(X)$

Definition 3.2. The type of *genuine unordered pairs* in X is the genuine orbit type of $\mathrm{EP}(X)$, $\mathrm{UP}(X) :\equiv \mathrm{EP}(X)_{\Sigma_2}$.

From our colimit calculation above, we get a pushout square:

$$\begin{array}{ccc} \mathrm{B}\Sigma_2 \times X & \xrightarrow{\mathrm{pr}_2} & X \\ \big\downarrow & \ulcorner & \big\downarrow{\scriptstyle s} \\ \mathrm{hUP}(X) & \xrightarrow[q]{} & \mathrm{UP}(X) \end{array}$$

The map on the left maps the pair (K, x) to (K, cst_x) i.e., the homotopy unordered pair corresponding to a singleton on x indexed by the two-element type K. We identify the points-to-pieces transformation on the right as the *singleton* map $s : X \to \mathrm{UP}(X)$ that maps $x : X$ to the unordered pair consisting of x with multiplicity 2, and we identify the map from the homotopy unordered pairs $q : \mathrm{hUP}(X) \to \mathrm{UP}(X)$ as a kind of quotient map. This is surjective, since the projection $\mathrm{B}\Sigma_2 \times X \to X$ is.

Note also the since $\mathrm{B}\Sigma_2$ and $\mathrm{hUP}(X)$ are essentially $\mathcal{U}$-small for $X : \mathcal{U}$, that $\mathrm{UP}(X)$ is essentially $\mathcal{U}$-small as well, since we assume $\mathcal{U}$ is closed under pushouts.

The following theorem was conjectured by Paolo Capriotti.

Theorem 3.3. *If X is a set, then $\mathrm{UP}(X)$ is again a set.*

To prove that $\mathrm{UP}(X)$ is a set, it suffices to show that all its loop spaces are contractible. So fix $x_0, x_1 : X$ representing in turn the ordered pair (x_0, x_1) and the corresponding homotopy and genuine unordered pairs, which we'll denote $w_0 :\equiv (\mathbf{2}, (x_0, x_1)) : \mathrm{hUP}(X)$ and $z_0 :\equiv q(w_0) : \mathrm{UP}(X)$, respectively.

Consider the proposition $P :\equiv (x_0 = x_1)$, and note that P is equivalent to the fiber of $\mathrm{B}\Sigma_2 \times X \to \mathrm{hUP}(X)$ at w_0. Our goal is to understand the identity types $z_0 = z$ in $\mathrm{UP}(X)$.

To this end, we follow an encode-decode strategy and introduce a type family $Q : \mathrm{UP}(X) \to \mathcal{U}$ meant to capture this identity type. Recall that the join $A * B$ of two types A and B is the pushout

$$\begin{array}{ccc} A \times B & \xrightarrow{\mathrm{pr}_2} & B \\ {\scriptstyle \mathrm{pr}_1}\big\downarrow & \ulcorner & \big\downarrow{\scriptstyle \mathrm{inr}} \\ A & \xrightarrow[\mathrm{inl}]{} & A * B. \end{array}$$

By pushout induction, we now set, for the point constructors:

$$\begin{aligned} Q(q(K, v)) &= (w_0 = (K, v)) * (P \times (v =_{\mathrm{El}\,K \to X} \mathrm{cst}_{x_0})) \\ Q(s(x)) &= P \times (x = x_0) \end{aligned}$$

The first case uses the join $*$ with the right-hand type a proposition, since X is a set. To complete the definition of Q we need to provide for each $K : \mathrm{B}\Sigma_2$ and each $x : X$ and equivalence $Q(q(K, \mathrm{cst}_x)) \simeq Q(s(x))$. Clearly, $Q(s(x))$ is a proposition (since X is a set). To show that $Q(q(K, \mathrm{cst}_x))$ is a proposition, we show that it is contractible assuming we have an element. Since being contractible is a proposition, for this it suffices to check the point constructors of the join. First assume we have an identification $w_0 = (K, \mathrm{cst}_x)$. Then we have $x = x_0 = x_1$, and the proposition $P \times (v = \mathrm{cst}_{x_0})$ holds, so the join is contractible. (Note $A * 1$ is always contractible.) The case where this proposition is inhabited is even more direct.

This analysis also shows that the propositions $Q(q(K, \mathrm{cst}_x))$ and $Q(s(x))$ are equivalent, as desired, thus completing the definition of Q.

Lemma 3.4. *The total type $(z : \mathrm{UP}(X)) \times Q(z)$ is contractible.*

Proof. Let us abbreviate $D :\equiv (z : \mathrm{UP}(X)) \times Q(z)$. By the flattening lemma for pushouts [11, Sec. 6.12], this type can be expressed as the pushout

$$\begin{array}{ccc} \mathrm{B}\Sigma_2 \times P & \xrightarrow{\mathrm{pr}_2} & P \\ {\scriptstyle g}\big\downarrow & \ulcorner & \big\downarrow{\scriptstyle \mathrm{inr}} \\ R & \xrightarrow[\mathrm{inl}]{} & D. \end{array}$$

where $R :\equiv (w : \mathrm{hUP}(X)) \times Q(q(w))$ and $g(K, p) :\equiv ((K, \mathrm{cst}_{x_0}), \mathrm{inr}(p, \mathrm{refl}))$. By the commutativity of Σ-types and colimits (here, joins), the type R is itself expressible as a pushout

$$\begin{array}{ccc} P & \longrightarrow & \mathrm{B}\Sigma_2 \times P \\ \big\downarrow & \ulcorner & \big\downarrow{\scriptstyle g} \\ 1 & \xrightarrow[\mathrm{pt}]{} & R, \end{array}$$

where the map $P \to \mathrm{B}\Sigma_2 \times P$ maps p to $(\mathbf{2}, p)$.

Define the map $h : R \to \mathrm{B}\Sigma_2$ by pushout induction, setting

$$h(g(K,p)) :\equiv K$$
$$h(\mathrm{pt}) :\equiv \mathbf{2}$$

and using the identity equivalence on $\mathbf{2}$ in the path constructor case. Note that h is an equivalence whenever P holds, in which case it's the inverse of g modulo the equivalence $\mathrm{B}\Sigma_2 \times P \simeq \mathrm{B}\Sigma_2$. In particular, we have $g(h(r), p) = r$ for any $r : R$ and $p : P$.

To prove that D is contractible, we show that $d = \mathrm{inl}(\mathrm{pt})$ for all $d : D$ by pushout induction.

- Suppose we have $p : P$ so $x_0 = x_1$. Then $\mathrm{glue}(\mathbf{2}, p)^{-1} : \mathrm{inr}(p) = \mathrm{inl}(\mathrm{pt})$.
- Suppose we have $r : R$. We construct an identification $\mathrm{inl}(r) = \mathrm{inl}(\mathrm{pt})$ by induction on r.
  - In the case $r \equiv \mathrm{pt}$ we take the reflexivity identification at $\mathrm{inl}(\mathrm{pt})$.
  - In the case $r \equiv g(K, p)$ we have the composite

$$\mathrm{inl}(g(K,p)) = \mathrm{inr}(p) = \mathrm{inl}(g(\mathbf{2},p)) = \mathrm{inl}(\mathrm{pt}).$$

  - These constructions agree under the assumption $p : P$
- By path algebra, the above constructions can be identified in the case $K : \mathrm{B}\Sigma_2$ and $p : P$. □

By the fundamental theorem of identity types [11, theorem 5.8.2], the identity types $z_0 = z$ in $\mathrm{UP}(X)$, when X is a set, are equivalent to $Q(z)$.

We can now finish the proof the $\mathrm{UP}(X)$ is a set by proving that $Q(z_0)$ is contractible. (We already know it's a set, since it's the join of a set with a proposition.) We take $\mathrm{inl}(\mathrm{refl}_{w_0})$ as the center of contraction. If we have an identification $w_0 = w_0$, then this consists of a bijection $e : \mathbf{2} \simeq \mathbf{2}$ together with $(x_0, x_1) \circ e =_{\mathbf{2}\to X} (x_0, x_1)$. If e is the identity, then we're done. Otherwise, e is the swapping bijection, and $x_0 = x_1$, so P holds and $Q(z_0)$ is contractible. In the other case (P holds and $(x_0, x_1) = \mathrm{cst}_{x_0}$), we directly conclude that $Q(z_0)$ is contractible.

Corollary 3.5. *If X is a set, then q exhibits* $\mathrm{UP}(X)$ *as the set truncation of* $\mathrm{hUP}(X)$*. Furthermore, the map from* $\mathrm{UP}(X)$ *to* $\mathrm{sUP}(X)$ *is an equivalence.*

4. The trouble with triples

The groupoid of objects of the orbit category Orb_{Σ_3} is

$$\begin{aligned}\mathrm{Orb}^{\mathrm{core}}_{\Sigma_3} &:= (X : \mathrm{B}\Sigma_3 \to \mathrm{Set}) \times \mathrm{isConn}_0((J : \mathrm{B}\Sigma_3) \times X\,J) \times \mathrm{hasDecEq}(X\,\mathrm{pt}) \\ &= (X : \mathrm{B}\Sigma_3 \to \mathcal{U}) \times \Big(\|X = \mathrm{Ord}\| + \|X = \mathrm{El}\| + \|X = \mathrm{Cyc}\| + \|X = \mathrm{Triv}\|\Big) \\ &= \mathrm{B}\Sigma_3 + (1 + \mathrm{B}\Sigma_2) + 1,\end{aligned}$$

where

$$\begin{aligned}\mathrm{Ord}(J) &= (\mathbf{3} = J) = (\text{the type of total orders on } J) \\ \mathrm{El}(J) &= J = (\text{the type of elements of the 3-element set } J) \\ \mathrm{Cyc}(J) &= (f : J \to J) \times \|(J, f) = (\mathbf{3}, \mathrm{succ})\| = (\text{the type of cyclic orders on } J) \\ \mathrm{Triv}(J) &= 1 = (\text{the unit type}).\end{aligned}$$

Every orbit corresponds to a gerbe (connected 1-type) Y and a 0-truncated (unpointed) map $Y \to \mathrm{B}\Sigma_3$. These gerbes are merely banded by either the trivial group, the cyclic group of order 2, the cyclic group of order 3, or the group Σ_3 itself. We'll write them also as $i_H : BH \to \mathrm{B}\Sigma_3$, where H is the corresponding subgroup-up-to-conjugation.

Suppose we have a homotopy Σ_3-type $A : \mathrm{B}\Sigma_3 \to \mathcal{U}$. Then for every orbit / subgroup-up-to-conjugation $i_H : BH \to \mathrm{B}\Sigma_3$, we get an induced action $A \circ i_H : BH \to \mathcal{U}$, and an induced map on homotopy fixed points $i_H^* : A^{\mathrm{h}\Sigma_3} \to A^{\mathrm{h}H}$.

For the unique subgroups-up-to-conjugation Σ_2 and Σ_3, that's all there is to it.

However, there is a Σ_2-action on the C_3 subgroup-up-to-conjugation (just coming from the automorphism group $\Sigma_2 \simeq \mathrm{Aut}(\mathrm{C}_3)$): For every $K : \mathrm{B}\Sigma_2$, we have the orbit $X_K\, J = (K = \mathrm{Cyc}(J))$ with corresponding subgroup-up-to-conjugation $i_K : \mathrm{BC}_3^{(K)} \to \mathrm{B}\Sigma_3$, meaning Σ_2 acts on the homotopy C_3-fixed points of A.

(Note also the degenerate situation of the trivial subgroups-up-to-conjugation 1, which carry a Σ_3-action. This gives the Σ_3-action on the homotopy e-fixed points $A^{\mathrm{h}e} = A^e$.)

A genuine Σ_3-type A now consists of

$$
\begin{aligned}
&A_0 : \mathrm{B}\Sigma_3 \to \mathcal{U}\\
&A_2 : A^{\mathrm{h}\Sigma_2} \to \mathcal{U}\\
&A_3 : A^{\mathrm{hC}_3} /\!\!/ \Sigma_2 \to \mathcal{U}\\
&A_\top : (v : A^{\mathrm{h}\Sigma_3}) \to (u : A_2\,(i_{\Sigma_2}^*\, v)) \to (t : (K : \mathrm{B}\Sigma_2) \to A_3\,(K, i_{\mathrm{C}_3^{(K)}}^*\, v)) \to \mathcal{U}
\end{aligned}
$$

Here, A_0 is the underlying homotopy Σ_3-action, while A_2 is the data of when a homotopy Σ_2-fixed point is genuine. Then A_3 is the data of when any point in the Σ_2-orbit type of homotopy C_3-fixed points is a genuine C_3-fixed point. An alternative type for A_3 would be the curried form

$$A_3' : (K : \mathrm{B}\Sigma_2) \to A^{\mathrm{hC}_3^{(K)}} \to \mathcal{U},$$

making it clear that Σ_2 acts on the genuine C_3-fixed points.

Finally, $A_\top$ is the data of when a homotopy Σ_3-fixed point is genuine, indexed by the data that the induced Σ_2- and $\mathrm{C}_3^{(K)}$-fixed points are all genuine.

Cohesive structure. The global sections of a genuine Σ_3-type A is the total type of $A_\top$, i.e.,

$$
\begin{aligned}
\Gamma\, A =&(v : A^{\mathrm{h}\Sigma_3}) \times (u : A_2\,(i_{\Sigma_2}^*\, v)) \times (t : (K : \mathrm{B}\Sigma_2) \to A_3\,(K, i_{\mathrm{C}_3^{(K)}}^*\, v))\\
&\times A_\top\, v\, u\, t.
\end{aligned}
$$

The constant genuine Σ_3-type ΔX on a type X has:

$$
\begin{aligned}
&\Delta X_0\, J &&= X\\
&\Delta X_2\, v &&= (x : X) \times (v = \mathrm{cst}_x)\\
&\Delta X_3\,(K, v) &&= (x : X) \times (v = \mathrm{cst}_x)\\
&\Delta X_\top\, v\, u\, t &&= (x : X) \times (p : v = \mathrm{cst}_x) \times (p_* u = \mathrm{mk}_2(x)) \times (p_* t = \mathrm{mk}_3(x)),
\end{aligned}
$$

where $\mathrm{mk}_2(x) :\equiv (x, \mathrm{refl})$ and $\mathrm{mk}_3(x)\, K :\equiv (x, \mathrm{refl})$.

Triples. For pairs, every homotopy fixed point of X^2 is genuine, since $(K : \mathrm{B}\Sigma_2) \to (K \to X) \simeq X$. For triples, there's trouble, however:

The homotopy C_3-fixed points of X^3 are already X, since $(J : \mathrm{B}\Sigma_3) \to \mathrm{Cyc}(J) \to (J \to X) \simeq X$, while the homotopy Σ_3-fixed points are $(J : \mathrm{B}\Sigma_3) \to J \to X \simeq$

$\mathrm{B}\Sigma_2 \to X$. Also the homotopy Σ_2-fixed points are troublesome: $(K : \mathrm{B}\Sigma_2) \to (K + 1 \to X) \simeq X \times (\mathrm{B}\Sigma_2 \to X)$

This means that $\mathrm{ET}(X)$, the type of genuine equivariant triples in X has:

$$\begin{aligned}
\mathrm{ET}(X)_0\, J &= (J \to X)\\
\mathrm{ET}(X)_2\, (x, v) &= (y : X) \times (v =_{\mathrm{B}\Sigma_2 \to X} \mathrm{cst}_x)\\
\mathrm{ET}(X)_3\, (K, v) &= 1\\
\mathrm{ET}(X)_\top\, v\, (y, h)\, t &= (k : v =_{(J:\mathrm{B}\Sigma_3) \to J \to X} \mathrm{cst}_h) \times ((K : \mathrm{B}\Sigma_2) \to ?)
\end{aligned}$$

5. Conclusion

We have defined the type of genuine unordered pairs $\mathrm{UP}(X)$ in any type X such that if X is a set, then so is $\mathrm{UP}(X)$, and so in this case $\mathrm{UP}(X)$ agrees with the set quotient of the set of ordered pairs in X modulo the natural equivalence relation.

We conjecture that more generally, $\mathrm{UP}(X)$ is an n-type whenever X is.

We have also defined the type of genuine unordered triples $\mathrm{UT}(X)$ in any type X. We conjecture that the corresponding fact, that $\mathrm{UT}(X)$ is a set when X is, is provable with a similar "elementary", but more complicated, proof.

At present, it seems as difficult to define the type of genuine unordered tuples, i.e., genuine multisets, in an arbitrary type, as it is to define the type of semi-simplicial types and thus solve the coherence problem. (The type of homotopy multisets in X is simply $(A : \mathrm{FinSet}) \times (\mathrm{El}\, A \to X)$.)

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