The $(\infty, 1)$-category of $(\infty, 1)$-categories in simplicial type theory

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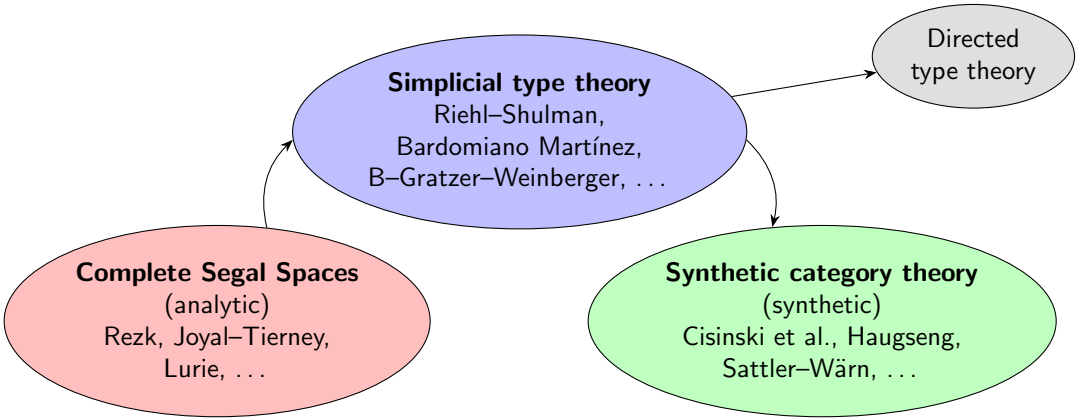
Introduction & Motivation

- In *Univalent Foundations*, done in Homotopy Type Theory, every type of mathematical object is represented by a type A, such that the identity type $a =_A b$ represents the natural notion of *identification* of objects $a, b : A$.
- Since every construction respects identity types, this makes all constructions invariant under identification/equivalence.
- For example, there is a 2-type of (univalent) 1-categories, $1\mathrm{Cat}$, such that $C =_{1\mathrm{Cat}} D$ is equivalent to the type of equivalences of categories $C \simeq D$.

However:

- In (synthetic) homotopy theory, we need the homotopical analog of 1-categories, $(\infty, 1)$-categories, where hom-*sets* are replaced by hom-*types*.
- We don't know how to define the type Cat, of $(\infty, 1)$-categories, following the *Structure Identity Principle*, and such that types & functions, spectra & maps, etc. are elements.
- This is the famous open problem of (semi-)simplicial types or infinitely coherent objects.
- Even if we had a solution, it might be very cumbersome to work with ...

Simplicial Type Theory as a Bridge



Simplicial type theory functions as a bridge, working with simplicial spaces synthetically to make the analytic theory of higher categories easier.

Simplicial type theory

In simplicial type theory [RS17] we interpret HoTT in simplicial spaces (or simplicial objects of any higher topos). These are probed by the simplices Δ^n.

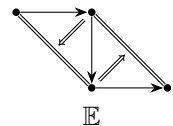
From an interval type $\mathbb{I}$ (a totally ordered set with distinct 0 and 1), we get the type of arrows $X^{\mathbb{I}}$ for any type X.

Definition X is *Segal* if $X^{\Delta^2} \to X^{\Lambda^2_1}$ is an equivalence.

$\Lambda^2_1 \hookrightarrow \Delta^2$



Definition A Segal type X is *Rezk* if $X^{\mathbb{E}} \to X$ is an equivalence, for $\mathbb{E}$ the "walking equivalence".



$\mathbb{E}$

The Rezk types are categories; Segal types are precategories. It's possible to do quite a lot of abstract category theory with these, incl. fibered category theory [BW23].

Proof assistant RZK [Kud23], formalization of a version of the Yoneda lemma [KRW04].

Modal simplicial type theory

But there aren't many *examples* of categories in bare simplicial type theory. In particular, we'd like the category of spaces $\mathcal{S}$ and the category of categories, Cat.

We define these using the [LOPS18] construction, but we need to be in a setting where $\mathbb{I}$ is *tiny*, which it isn't in simplicial spaces, but it is in cubical spaces: presheaves on the full subcategory on $\{(0 < 1)^n\}$ of Cat.

With MTT [GKNB21; GKNB20], we add modalities

- idempotent comonad $\flat$ (global/objectwise) with right adjoint $\sharp$,
- involutive op (opposite),
- tw (twisted arrows) with 2-cells $\mathrm{tw} \to \mathrm{id}$, $\mathrm{tw} \to \mathrm{op}$,
- equations $\flat = \flat\flat = \flat\sharp = \flat\mathrm{op} = \mathrm{tw}\flat$, ...

Axioms include: $\mathbb{I}$ is a tiny bounded distributive lattice with $\langle \flat \mid 2 \rangle \simeq \langle \flat \mid \mathbb{I} \rangle$ and de Morgan duality $\langle \mathrm{op} \mid \mathbb{I} \rangle \simeq \mathbb{I}$.

$\mathbb{I}$ is tiny if $\mathbb{I} \to _$ has a right adjoint.

See also Riley's [Ril24], and [WL20].

Category theory in modal simplicial type theory

We can use the modalities to state and prove standard facts in category theory:

- If $C, D :_\flat \mathcal{U}$ are categories and $f, g :_\flat C \to D$, then a natural transformation $\alpha :_\flat \hom(f, g)$ is invertible if and only if for all $c :_\flat C$ the map $\alpha(c) : \hom(f\,c, g\,c)$ is invertible.
- Similarly, f is invertible iff f is fully faithful and essentially surjective on objects, i.e., for $\flat$-elements.

An adjunction $C \underset{g}{\overset{f}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} D$ is given by a family of equivalences $\alpha : \prod_{c:C} \prod_{d:D} \hom(f\,c, d) \simeq \hom(f, g\,d)$.

- If $C, D :_\flat \mathcal{U}$ are categories, then $f :_\flat C \to D$ is a left adjoint iff for all $d :_\flat D$ there exists $c :_\flat C$ and $\epsilon :_\flat \hom(f(c), d)$ such that the following is an equivalence for all $c' :_\flat C$: $\epsilon_* \circ f : \hom(c', c) \to \hom(f(c'), d)$.

Triangulated type theory

Cubical spaces contain simplicial spaces as a subtopos [SW21], which we can define as the lex modality $\boxslash$, nullification at the family of propositions:

$$\Phi : \mathbb{I} \times \mathbb{I} \to \mathsf{Prop}, \qquad \Phi(i,j) = (i \leq j) \vee (j \leq i)$$

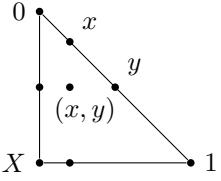
The $\boxslash$-modal types are the simplicial spaces. We add as axioms that cubes are projective generators [MR23].

Further, we use synthetic quasicoherence [Ble23], which relies on the fact that $\mathbb{I}$, qua cubical space, is simplicial, see also [Spi16].

The interval $\mathbb{I}$ is a distributive lattice with $0, 1$. Define an $\mathbb{I}$-algebra to be a distributive lattice R with $0, 1$ and a structure-preserving map $\mathbb{I} \to R$.

Axiom (Duality) For any finitely presented $\mathbb{I}$-algebra R, the map $R \to (\hom_{\mathbb{I}}(R, \mathbb{I}) \to \mathbb{I})$ is an equivalence.





Cor. (Phoa's principle) We have equivalences

$$\Delta^2 \simeq \mathbb{I}[X] \simeq (\mathbb{I} \to \mathbb{I}).$$

(Amazingly) Covariant Families

Consider the notion of covariant families:

$$\mathrm{Cov} : \mathcal{U}^X \to \mathsf{Prop}$$

expresses that a family is right orthogonal to $\{0\} \to \mathbb{I}$.

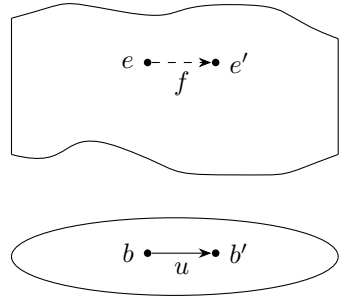
Theorem $\mathcal{S} \triangleq \mathcal{U}_{\mathsf{ACov}} \triangleq \mathcal{U} \times_{\mathsf{Prop}_{\mathbb{I}}} \mathbf{1}_{\mathbb{I}}$ classifies amazingly covariant families: for all $X :_\flat \mathcal{U}$, the canonical map

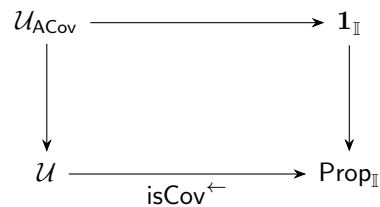
$$\langle \flat \mid X \to \mathcal{S} \rangle \to \left\langle \flat \mid \textstyle\sum_{A : X \to \mathcal{U}} \mathsf{isCov}(A) \right\rangle$$

is an equivalence.

Closure properties $\mathcal{S}$:

- contains global discrete types,
- is closed under Σ and identity types,
- is closed under finite limits and colimits.





Directed Univalent Universe of Covariant Families

To prove directed univalence we define directed glue types, given a type X and a family $A : X \to \mathcal{S}$:

$\mathsf{Gl}\, A : X \times \mathbb{I} \to \mathcal{U}_{\triangleright}$
$\mathsf{Gl}\, A\, x\, i = (i = 0) \to A\, x$

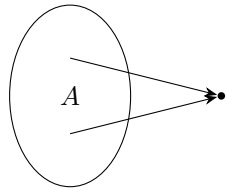
Theorem $\mathsf{Gl}\, A :_g X \times \mathbb{I} \to \mathcal{S}$.
This uses Phoa's principle.

Given $E_0, E_1 : \mathcal{S}$ with $f : E_0 \to E_1$ we get a type family

$$\textstyle\sum_{e_1 : E_1} \mathsf{Gl}\, f^{-1}(e_1) : \mathbb{I} \to \mathcal{S}$$

This gives us straightening.

Theorem (Directed univalence)
$S : (E_0 \to E_1) \to \hom_{\mathcal{S}}(E_0, E_1)$ is an equivalence.



Corollary $\mathcal{S}$ is Segal and Rezk, hence a category.

Theorem $\mathcal{S}$ is simplicial.
Currently, for this we axiomatize that $\triangleright A$ has the same n-simplices as A.

Directed structure identity principle

As a consequence of directed univalence we can construct many other categories with the expected morphisms:

- $(\infty, 1)$-category of pointed spaces, $\mathcal{S}_\star = \sum_{X:\mathcal{S}} X$,
- 1-category of finite sets,
- 1-categories of usual algebraic structures,
- 1-category of posets, ω, Δ,
- $(\infty, 1)$-category of spectra $\mathsf{Sp} = \varprojlim (\mathcal{S}_\star \leftarrow \mathcal{S}_\star \leftarrow \dots)$,
- $(\infty, 1)$-category of spaces with an endomorphism,
- ...

We can prove Sp is cocomplete and stable.

(Amazingly) Cocartesian Families

To tackle the category of categories Cat, we need to classify cocartesian families. An arrow $f : \mathbb{I} \to \widetilde{A}$ in a family $A : X \to \mathcal{U}$ is *cocartesian* if diagrams of the following shape have a unique lift whenever $\Delta^{\{\bar{0}\to\bar{1}\}} \to \Delta^2 \to \widetilde{A}$ is f:

$$\begin{array}{ccc} \Lambda^2_0 & \longrightarrow & \widetilde{A} \\ \downarrow & & \downarrow \\ \Delta^2 & \longrightarrow & X \end{array}$$

A family $A : X \to \mathcal{U}$ is *cocartesian* if it is iso-inner, each $A(x)$ is simplicial, and the following holds:
$\prod_{u:\mathbb{I}\to X} \prod_{a:A(u\,0)} \sum_{f:\hom^A_u(a,\bullet)} \mathsf{isCocartArr}_A(f)$

In particular, in a cocartesian family, every arrow factors uniquely as a cocartesian arrow followed by a vertical arrow. A cocartesian family has functorial cocartesian transport maps.

Thanks to an observation by Rob Schellingerhout, naively locally cocartesian families, i.e., those whose restriction $A \circ f$ for any $f : \mathbb{I} \to X$ is cocartesian, are *already* cocartesian.

To get the real notion of locally cocartesian family, we need to restrict to $f :_\flat \mathbb{I} \to X$.

(Amazingly) Cocartesian Families

Theorem $\mathsf{Cat} \triangleq \sum_{A:\mathcal{U}_{\triangleright}} \mathsf{isRezk}\, A \times \mathsf{aIsInner}\, A \times \mathsf{aHasNLCCLifts}\, A$ is the base of the universal cocartesian family, i.e., for all $X :_\flat \mathcal{U}$, the canonical map

$$\langle \flat \mid X \to \mathsf{Cat} \rangle \to \left\langle \flat \mid \textstyle\sum_{A:X\to\mathcal{U}} \mathsf{isCocart}(A) \right\rangle$$

is an equivalence.

Directed univalence If A, B are $\flat$-elements of Cat, then there is a equivalence $\mathsf{dua} : \hom_{\mathsf{Cat}}(A,B) \simeq \langle \flat \mid A \to B \rangle$. In general, we get an equivalence between $\langle \flat \mid X \times \mathbb{I} \to \mathsf{Cat} \rangle$ and $\left\langle \flat \mid \sum_{A_0,A_1:\mathsf{Cat}^X} A_0 \to^{\mathsf{cc}} A_1 \right\rangle$.

With a bit more work, we characterize cocartesian families $A :_\flat X \times \Delta^2 \to \mathcal{U}$, allowing us to prove that Cat is Segal and Rezk.

The hardest part is then to show Cat is simplicial, which can be done by giving a suitable map back $\triangleright\mathsf{Cat} \to \mathsf{Cat}$.

Examples

- The map $i : \mathcal{S} \to \mathrm{Cat}$ is fully faithful and possesses both left and right adjoints.
- The inclusion $\mathrm{Cat}_n \hookrightarrow \mathrm{Cat}$ has a left adjoint.
- The category of monoidal categories MCat is the full subcategory of $\langle \mathrm{op} \mid \Delta \rangle \to \mathrm{Cat}$ spanned by monoidal categories, those $C^{\otimes} : \mathrm{Cat}^{\langle \mathrm{op} \mid \Delta \rangle}$ for which $(\rho_n^1, \dots, \rho_n^n) : C^{\otimes}([n]) \to C^{\otimes}([1]) \times \cdots \times C^{\otimes}([1])$ is an equivalence for all n.
- The category of E_1-monoids is similarly a subcategory of $\mathcal{S}^{\langle \mathrm{op} \mid \Delta \rangle}$.
- We get a simple form of algebraic K-theory as a composite:

$$k : \mathrm{MCat} \to \mathrm{Mon} \to \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*$$

  (Use that $(-)^{\simeq} : \mathrm{Cat} \to \mathcal{S}$ is a right adjoint, inducing a functor $\mathrm{MCat} \to \mathrm{Mon}$.)
- $\sum_{A:\mathrm{Cat}} A^C$ is the lax slice category.

Thank you

References I

[Ble23] Ingo Blechschmidt. *A general Nullstellensatz for generalized spaces*. Draft. 2023. URL: https://rawgit.com/iblech/internal-methods/master/paper-qcoh.pdf (cit. on p. 7).

[BW23] Ulrik Buchholtz and Jonathan Weinberger. "Synthetic fibered $(\infty, 1)$-category theory". In: *Higher Structures* 7 (1 2023), pp. 74–165. DOI: 10.21136/HS.2023.04 (cit. on p. 4).

[GKNB21] Daniel Gratzer, G. A. Kavvos, Andreas Nuyts, and Lars Birkedal. "Multimodal Dependent Type Theory". In: *Logical Methods in Computer Science* Volume 17, Issue 3 (July 2021). DOI: 10.46298/lmcs-17(3:11)2021 (cit. on p. 5).

[GKNB20] Daniel Gratzer, G.A. Kavvos, Andreas Nuyts, and Lars Birkedal. "Multimodal Dependent Type Theory". In: *Proceedings of the 35th Annual ACM/IEEE Symposium on Logic in Computer Science*. LICS '20. ACM, 2020. DOI: 10.1145/3373718.3394736 (cit. on p. 5).

[GWB24] Daniel Gratzer, Jonathan Weinberger, and Ulrik Buchholtz. *Directed univalence in simplicial homotopy type theory*. 2024. arXiv: 2407.09146.

References II

[GWB25] Daniel Gratzer, Jonathan Weinberger, and Ulrik Buchholtz. "The Yoneda embedding in simplicial type theory". In: *40th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2025, Singapore, June 23-26, 2025*. IEEE, 2025, pp. 127–142. DOI: 10.1109/LICS65433.2025.00017.

[GWB26] Daniel Gratzer, Jonathan Weinberger, and Ulrik Buchholtz. *The ∞-category of ∞-categories in simplicial type theory*. 2026. arXiv: 2602.02218.

[Kud23] Nikolai Kudasov. RZK. An experimental proof assistant based on a type theory for synthetic ∞-categories. 2023. URL: https://github.com/rzk-lang/rzk (cit. on p. 4).

[KRW04] Nikolai Kudasov, Emily Riehl, and Jonathan Weinberger. "Formalizing the ∞-Categorical Yoneda Lemma". In: *Proceedings of the 13th ACM SIGPLAN International Conference on Certified Programs and Proofs*. 2004, pp. 274–290. DOI: 10.1145/3636501.3636945 (cit. on p. 4).

References III

[LOPS18] Daniel R. Licata, Ian Orton, Andrew M. Pitts, and Bas Spitters. "Internal Universes in Models of Homotopy Type Theory". In: *3rd International Conference on Formal Structures for Computation and Deduction (FSCD 2018)*. Ed. by Hélène Kirchner. Vol. 108. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2018, 22:1–22:17. DOI: 10.4230/LIPIcs.FSCD.2018.22 (cit. on p. 5).

[MR23] David Jaz Myers and Mitchell Riley. *Commuting Cohesions*. 2023. arXiv: 2301.13780 [math.CT] (cit. on p. 7).

[RS17] Emily Riehl and Michael Shulman. "A type theory for synthetic ∞-categories". In: *Higher Structures* 1 (1 2017), pp. 147–224. DOI: 10.21136/HS.2017.06 (cit. on p. 4).

[Ril24] Mitchell Riley. *A Type Theory with a Tiny Object*. 2024. arXiv: 2403.01939 [math.CT] (cit. on p. 5).

[Spi16] Bas Spitters. *Cubical sets and the topological topos*. 2016. arXiv: 1610.05270 [cs.LO] (cit. on p. 7).

References IV

[SW21] Thomas Streicher and Jonathan Weinberger. "Simplicial sets inside cubical sets". In: *Theory and Application of Categories* 37.10 (2021), pp. 276–286 (cit. on p. 7).

[WL20] Matthew Z. Weaver and Daniel R. Licata. "A Constructive Model of Directed Univalence in Bicubical Sets". In: *Proceedings of the 35th Annual ACM/IEEE Symposium on Logic in Computer Science*. LICS '20. ACM, July 2020. DOI: 10.1145/3373718.3394794 (cit. on p. 5).