Coinductive definitions and constructive mathematics

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Overview and motivation

I got interested in coinductive definitions since I needed one:
the perfect core of a group, and the (fiber of the) plus construction of a pointed type.

These are examples of non-fundamental coinductive definitions (in Kleene's sense).

I'll explain the framework in which I work: univalent, constructive mathematics.

We'll look at the above examples, and many others, of coinductive definitions.

We'll see that fundamental coinductive definitions are definable from natural numbers (Ahrens, Capriotti, and Spadotti 2015), but it's not so clear what happens with non-fundamental ones.

I'll review some axioms (WISC and RRS) which have some chance of being constructively & predicatively acceptable on the basis of invariance, then conclude with some open questions.

Univalent mathematics

Univalent mathematics is extensional mathematics: We are only satisfied with the definition of a type A of mathematical objects if the identity type $a =_A b$ expresses the correct notion of identifications of a with a'. Structure Identity Principle (SIP)

- Any two elements of a proposition can be identified. (This defines the notion of proposition.)
- A *set* is a type A for which each identity type $a =_A a'$ is a proposition.
- ... continuing, we get notions of groupoid/1-type, 2-groupoid/2-type, ..., n-type, ...
- The identity type $A =_{\mathcal{U}} B$ is equivalent to the type of equivalences $A \simeq B$ (univalence)
- ... from which follows function extensionality and
- the SIP for naturally defined types of (generalized) algebraic models (groups, rings, categories, etc.).
- In particular, $A =_{\mathrm{Set}} B$ is equivalent to the set of isomorphisms $A \cong B$.
- For $x, y : \mathbb{R}$, we should be able to identify $x =_{\mathbb{R}} y$ with $\neg(x < y \vee y < x)$.

Constructive mathematics

I take constructive mathematics to entail that we reason by constructions of types and their elements.

The univalence axiom as an addition to Martin-Löf type theory is not immediately constructive in this sense, but we now have cubical type theories (with normalization) in which univalence can be constructed, so we see that univalence is constructive. There's also work towards higher observational type theory (Altenkirch, Chamoun, Kaposi, and Shulman 2024), which likewise promises constructions suitable for univalent mathematics.

A key, though not defining, feature of constructive mathematics is that we have canonicity and that its means of construction are preserved by passage to sheaf topos and realizability models.

For HoTT, we can so far only pass directly to a subtopos model, and certain internal presheaf models. (Major obstacles remain in the univalent model theory of univalent theories.)

From a classical base, we can pass to any higher sheaf topos (Shulman 2019) and we have cubical realizability models (Uemura 2019). The latter *doesn't validate propositional resizing* (small power sets).

Epimorphisms in univalent mathematics

A map $f : A \to B$ is an *epimorphism* if for every type X, the precomposition map

$$(B \to X) \xrightarrow{f^*} (A \to X)$$

is an embedding. (Extensions along f are unique.)

A type is *acyclic* if its suspension is contractible; a map is *acyclic* if all its fibers acyclic.

Theorem (Buchholtz, de Jong, and Rijke (2024))

For a map $f : A \to B$, the following are equivalent:

- *f is an epimorphism,*
- *f is acyclic,*
- *f is balanced, i.e., for every surjection $g : X \to B$, the pullback square of f, g is also a pushout square.*

The perfect core

We would like to construct the factorization of any map $f : A \to B$ as an epimorphism followed by a hypoabelian map, where a map is hypoabelian if the fundamental groups of its fibers have no perfect subgroups. (G is perfect if the abelianization $G^{\mathrm{ab}} = G/[G, G]$ is trivial.)

So a first step is to construct the *perfect core*: the largest perfect subgroup of a group. Classically, it's the intersection of the transfinite derived series, obtained by iterating $H \mapsto H' = [H, H]$ (commutator subgroup):

$$G = G^{(0)} \supseteq G' = G^{(1)} \supseteq \cdots \supseteq G^{(\alpha)} \supseteq \cdots$$

This is a *non-fundamental* coinductive definition: We take the greatest fixed point of the finitely accessible operator $\Phi : (G \to \mathrm{Prop}) \to (G \to \mathrm{Prop})$ taking a subset X to the subgroup generated by commutators $[x, y] = xyx^{-1}y^{-1}$ for $x, y \in X$.

Theorem (Hartley (1965) and Maltsev (1949))

Any ordinal α can be the stabilization ordinal of the transfinite derived series of a group G.

The plus construction

Quillen's plus construction gives the factorization $X \to X^+ \to 1$ of the terminal projection as an acyclic map followed by a hypoabelian map (using the axiom of choice and Whitehead's principle). This is important for algebraic K-theory as, or a ring R, the homotopy groups of $\mathrm{BGL}(R)^+$ form the higher K-groups of R.

For a pointed type X, the fiber of $X \to X^+$ is the *largest* acyclic type with a map to X. Use balance property:

$$\begin{array}{ccc} F & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & X^+ \end{array}$$

Conjecture: X^+ is the terminal coalgebra of the operation $Y \mapsto \mathrm{Fib}(Y \to \Omega\Sigma Y)$ on pointed types Y over X.

Other examples of coinductive definitions

In general, a coinductively defined type C is determined by destructors; detailing the observations that are possible on elements of C:

- Function types $\prod_{x:A} B(x)$: the observations on f are the applications $f(a)$ for $a : A$.
- Conatural numbers $\mathbb{N}_\infty$: if $x : \mathbb{N}_\infty$ then either $x = 0$ or $x = \mathrm{suc}(y)$ for some $y : \mathbb{N}_\infty$.
- Streams over A, A^ω: if $x : A^\omega$, then we have $\mathrm{hd}(x) : A$ and $\mathrm{tl}(x) : A^\omega$.
- Non-wellfounded trees, $M_{x:A}B(x)$: if $t : M_{x:A}B(x)$, then $\mathrm{shape}(t) : A$ and $\mathrm{sub}(t)(y) : M_{x:A}B(x)$ for each $y : B(\mathrm{shape}(t))$.
- Coinductive unit interval (Freyd), $0, 1 : [0,1]_{\mathrm{F}}$: If $x : [0,1]_{\mathrm{F}}$, then $\mathrm{split}(x) : [0,1]_{\mathrm{F}} \vee [0,1]_{\mathrm{F}}$, the pushout of the $1, 0$ span $[0,1]_{\mathrm{F}} \leftarrow 1 \rightarrow [0,1]_{\mathrm{F}}$, together with $0 \neq 1$.

Application include:

- Modeling behavior of physical/computational systems (Jacobs/Rutten/...).
- Non-wellfounded set theory (Aczel).
- Modal logic.

Polynomial functors

The polynomial functors between categories of families of types are those generated by Σ, Π and substitution. They have the normal form

$$P : (I \to \mathcal{U}) \to (J \to \mathcal{U}), \quad P\ X\ j = \sum_{b:B(j)} \prod_{e:E(j,b)} X(\mathrm{tgt}(e))$$

given by a diagram $I \leftarrow E \to B \to J$.

An analytic functor is one for which all fibers of $E \to B$ are finite sets. (Infinite sums of finite monomials.)

The indexed W-types are the initial algebras for polynomial endofunctors on $I \to \mathcal{U}$.

Fundamental coinductive types

Let's take as *fundamental coinductive types* the terminal coalgebras for polynomial endofunctors (M-types).

Theorem (Ahrens, Capriotti, and Spadotti (2015))

We can construct indexed M-types from the natural numbers.

Proof.

Take the limit of ω-sequence

$$\cdots \to P^2(I) \to P(I) \to I.$$

This gives the terminal coalgebra, since P is ω-continuous. □

(A similar observation predates univalent mathematics.)

Notice that no strength at all is needed, in contrast to W-types.

RRS and Coinduction

The perfect core is not an instance of this construction, since we want a greatest fixed point of a truncated polynomial operator $(G \to \mathrm{Prop}) \to (G \to \mathrm{Prop})$, $X \mapsto (x \mapsto \|P\ X\ x\|)$. This is no longer ω-continuous.

It's easy to see, however, since P is analytic (finitary), that if we have dependent choice, then $x \mapsto \|M_P(x)\|$ is the greatest fixed point.

Aczel (2008) isolated a weaker principle, the Relation Reflection Scheme, that suffices:

- (RRS) For a large total relation $R \subseteq X \times X$, if $x_0 : X$, then there exists a small subset $a \subseteq X$ with $x \in a$ such that $\forall x \in a. \exists y \in a. R(x, y)$.

Aczel also showed that RRS is stable under passage to sheaves on complete Heyting algebras. Ziegler (2008, 2012) showed more generally that RRS is preserved by passage to sheaves on applicative topologies. (A common generalization of realizability and sheaf 1-toposes.)

Conjecture: RRS is stable under passage to higher toposes.

Comparison with induction: WISC and QITs

Blass (1983) gave an example of an infinitary algebraic theory for which ZF doesn't derive the existence of an initial algebra (relative to the consistency of ZFC + "strongly compact cardinals are cofinal in the ordinals").

Lumsdaine and Shulman (2019) then showed that this corresponds to a HIT that cannot be shown to exist in ZF.

However, with the axiom WISC (weakly initial set of covers), this, and many other quotient inductive types, are constructible (Fiore, Pitts, and Steenkamp 2022). Are there still uses of WISC that are not covered by HITs?

Conclusions

- What counts as constructive? Is canonicity plus invariance under sheaf and realizability models sufficient?
- Are there constructions of WISC + RRS?
- Is the perfect core definable in MLTT+UA+HITs?
- In Higher Observational Type Theory, coinductive definitions play a key role in defining observational identity types.
- Semisimplicial types have a coinductive definition in displayed type theory Kolomatskaia and Shulman (2024).
- Higher topos theory rhymes with predicativity, but are there mathematical applications of predicativity? (New model constructions?)

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